ANALYSIS OF THE DPG METHOD FOR THE POISSON EQUATION

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Outline of Presentation

- Abstract B^3 framework.

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- Proof of well-posedness for the DPG formulation in multidimensions.

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- Numerical experiments.

Petrov–Galerkin Method with Optimal Test Functions Abstract B^3 Framework (Repetitio Mater Studiorum Est)

Abstract Variational Problem

$$\begin{cases} u \in U \\ b(u,v) = l(v) \quad \forall v \in V \end{cases} \Leftrightarrow \begin{array}{l} Bu = l \quad B : U \to V' \\ < Bu, v >= b(u,v) \, \forall v \in V \end{array}$$

where

- U, V are Hilbert spaces,
- $b(u,v)$ is a continuous bilinear form on $U \times V$,

  $$|b(u,v)| \leq M \|u\|_U \, \|v\|_V$$

  that satisfies the inf-sup condition ($\Leftrightarrow B$ is bounded below),

  $$\inf_{\|u\|_U=1} \sup_{\|v\|_V=1} |b(u,v)| =: \gamma > 0$$

- $l \in V'$ represents the load and satisfies the compatibility condition $l(v) = 0, \forall v \in V_0$ where

  $$V_0 := \{v \in V \; : \; b(u,v) = 0 \quad \forall u \in U\}$$

Energy Norm

Banach Closed Range Theorem implies that there exists a unique solution u that depends continuously upon the data, $\|u\| \leq \frac{1}{\gamma}\|l\|_{V'}$. The supremum in the inf-sup condition defines an equivalent, problem-dependent *energy (residual) norm*,

$$\|u\|_E := \sup_{\|v\|=1} |b(u,v)| = \|Bu\|_{V'}$$

For the energy norm, $M = \gamma = 1$. Recalling that the Riesz operator is an isometry form V into V', we may characterize the energy norm in an equivalent way as

$$\|u\|_E = \|v_u\|_V$$

where v_u is the solution of the variational problem,

$$\begin{cases} v_u \in V \\ (v_u, \delta v)_V = b(u, \delta v) \quad \forall \delta v \in V \end{cases}$$

Optimal Test Functions

Select your favorite trial basis functions: $e_j,\ j = 1, \dots, N$. For each function e_j, introduce a corresponding *optimal test (basis) function* $\bar{e}_j \in V$ that realizes the supremum,

$$|b(e_j, \bar{e}_j)| = \sup_{\|v\|_V = 1} |b(e_j, v)|$$

i.e. it solves the variational problem,

$$\begin{cases} \bar{e}_j \in V \\ (\bar{e}_j, \delta v)_V = b(e_j, \delta v) \quad \forall \delta v \in V \end{cases}$$

Define the discrete test space as
$\bar{V}_{hp} := \text{span}\{\bar{e}_j,\ j = 1, \dots, N\} \subset V$. It follows from the construction of the optimal test functions that the *discrete* inf-sup constant

$$\inf_{\|u_{hp}\|_E = 1} \sup_{\|v_{hp}\| = 1} |b(u_{hp}, v_{hp})| = 1$$

The Best Approximation

Consequently, Babuška's Theorem

$$\|u - u_{hp}\|_E \leq \frac{M}{\gamma_{hp}} \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E$$

implies that

$$\|u - u_{hp}\|_E \leq \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E$$

i.e., the method delivers the *best approximation error* in the energy norm.

Stiffness Matrix Is Symmetric and Positive Definite

$$b(e_i, \bar{e}_j) = (\bar{e}_i, \bar{e}_j)_V = (\bar{e}_j, \bar{e}_i)_V = b(e_j, \bar{e}_i)$$

Energy Norm of FE Error $e_{hp} = u - u_{hp}$

can be computed *without* knowing the exact solution.

$$\left\{ \begin{array}{l} v_{e_{hp}} \in V \\ (v_{e_{hp}}, \delta v)_V = b(u - u_{hp}, \delta v) = l(\delta v) - b(u_{hp}, \delta v) \quad \forall \delta v \in V \end{array} \right.$$

We have then

$$\|e_{hp}\|_E = \|v_{e_{hp}}\|_V$$

We shall call $v_{e_{hp}}$ *the error representation function*

Note: No need for an a-posteriori error estimation.

Relation with Least Squares

Rewrite the variational problem in the operator form:

$$Bu = l, \quad B : U \to V', \quad < Bu, v >= b(u, v)$$

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Apply the least squares method

$$\| R_V^{-1} B u_{hp} - R_V^{-1} l \|_V \to \text{min}$$

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$$\|R_V^{-1} Bu_{hp} - R_V^{-1} l\|_V \to \text{min}$$

This is exactly our DPG method

Optimal Test Norm

Q: Can we select the norm in the test space in such a way that the corresponding energy norm coincides with the original norm (of choice) in U ?

Optimal test norm

A: Yes! Choose:

$$\|v\|_V = \sup_{u \in U} \frac{|b(u,v)|}{\|u\|_U}$$

(under assumption that

$$V_0 = \{v \in V \ : \ b(u,v) = 0 \quad \forall u \in U\}$$

is trivial)

Convergence Analysis in Multidimensions

Poisson Problem

$$\begin{cases} u &= u_0 \quad \text{on } \partial\Omega \\ -\boldsymbol{\nabla} \cdot (\boldsymbol{\alpha} \boldsymbol{\nabla} u) + \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u &= f \quad \text{in } \Omega \end{cases}$$

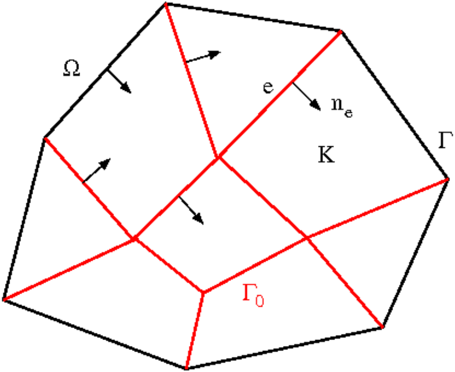
For a moment $\boldsymbol{\beta} = \mathbf{0}$.

Poisson Problem

First order system:

$$\begin{cases} \boldsymbol{\alpha}^{-1}\boldsymbol{\sigma} - \boldsymbol{\nabla} u &= \mathbf{0} \quad \text{in } \Omega \\ \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} &= f \quad \text{in } \Omega \\ u &= u_0 \quad \text{on } \partial\Omega \end{cases}$$

DPG Method



Elements: K
Edges: e
Skeleton: $\Gamma_h = \bigcup_K \partial K$
Internal skeleton: $\Gamma_h^0 = \Gamma_h - \partial\Omega$

DPG Method

Take an element K. Multiply the equations with test functions $\boldsymbol{\tau} \in \boldsymbol{H}(\text{div}, K), v \in H^1(K)$:

$$\begin{cases} (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} - (\boldsymbol{\nabla} u) \cdot \boldsymbol{\tau} &= 0 \\ (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) v &= fv \end{cases}$$

DPG Method

Integrate over the element K:

$$\begin{cases} \int_K (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} - \int_K (\boldsymbol{\nabla} u) \cdot \boldsymbol{\tau} & = 0 \\ \int_K (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) v & = \int_K f v \end{cases}$$

DPG Method

Integrate by parts (relax) *both* equations:

$$\begin{cases} \int_K (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} + \int_K u \, \mathrm{div}\boldsymbol{\tau} - \int_{\partial K} u \, \tau_n & = 0 \\ \qquad - \int_K \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v + \int_{\partial K} q \, \mathrm{sgn}(\boldsymbol{n}) v & = \int_K f v \end{cases}$$

where $q = \boldsymbol{\sigma}\boldsymbol{n}_e$ and

$$\mathrm{sgn}(\boldsymbol{n}) = \begin{cases} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{cases}$$

DPG Method

Declare fluxes to be independent unknowns:

$$\left\{ \begin{array}{rl} \int_K (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} + \int_K u \, \mathrm{div}\boldsymbol{\tau} - \int_{\partial K} \hat{u} \, \tau_n & = 0 \\ - \int_K \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v + \int_{\partial K} \hat{q} \, \mathrm{sgn}(\boldsymbol{n}) v & = \int_K f v \end{array} \right.$$

where $q = \boldsymbol{\sigma} \boldsymbol{n}_e$ and

$$\mathrm{sgn}(\boldsymbol{n}) = \left\{ \begin{array}{rl} 1 & \text{if } \boldsymbol{n} = \boldsymbol{n}_e \\ -1 & \text{if } \boldsymbol{n} = -\boldsymbol{n}_e \end{array} \right.$$

DPG Method

Use BCs to eliminate known fluxes

$$\left\{ \begin{array}{rl} \int_K (\boldsymbol{\alpha}^{-1}\boldsymbol{\sigma}) \cdot \boldsymbol{\tau} + \int_K u \, \mathrm{div}\boldsymbol{\tau} - \int_{\partial K - \partial\Omega} \hat{u} \, \tau_n & = + \int_{\partial K \cap \partial\Omega} u_0 \, \tau_n \\ - \int_K \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} v + \int_{\partial K} \hat{q} \, \mathrm{sgn}(\boldsymbol{n}) v & = \int_K f v \end{array} \right.$$

Trace and Flux Spaces

$$\begin{aligned}
\Gamma_h &:= \bigcup_K \partial K \quad \text{(skeleton)} \\
\Gamma_h^0 &:= \Gamma_h - \partial\Omega \quad \text{(internal skeleton)} \\
\tilde{H}^{1/2}(\Gamma_h^0) &:= \{V|_{\Gamma_h^0} \,:\, V \in H_0^1(\Omega)\} \\
&\quad \text{with the minimum extension norm:} \\
\|v\|_{\tilde{H}^{1/2}(\Gamma_h^0)} &:= \inf\{\|V\|_{H^1} \,:\, V|_{\Gamma_h^0} = v\} \\
H^{-1/2}(\Gamma_h) &:= \{\sigma_n|_{\Gamma_h} \,:\, \boldsymbol{\sigma} \in \boldsymbol{H}(\text{div},\Omega)\} \\
&\quad \text{with the minimum extension norm:} \\
\|\sigma_n\|_{H^{-1/2}(\Gamma_h)} &:= \inf\{\|\boldsymbol{\sigma}\|_{H(\text{div},\Omega)} \,:\, \boldsymbol{\sigma}\boldsymbol{n}|_{\Gamma_h} = \sigma_n\}
\end{aligned}$$

Functional Setting

Group variables:
Solution $\boldsymbol{U} = (u, \boldsymbol{\sigma}, \hat{u}, \hat{q})$:

$$
\begin{aligned}
&u, \sigma_1, \sigma_2 \in L^2(\Omega_h) \\
&\hat{u} \in \tilde{H}^{1/2}(\Gamma_h^0) \\
&\hat{q} \in H^{-1/2}(\Gamma_h)
\end{aligned}
$$

Test function $\boldsymbol{V} = (\boldsymbol{\tau}, v)$:

$$
\begin{aligned}
&\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega_h) \\
&v \in H^1(\Omega_h)
\end{aligned}
$$

Variational problem:

$$
b(\boldsymbol{U}, \boldsymbol{V}) = l(\boldsymbol{V}), \quad \forall \boldsymbol{V}
$$

Simple facts

- Form b is continuous
- $b(U,V) = 0, \forall V$ implies $U = 0$.

In operator terms,

$$b(U,V) =< BU, V >=< U, B^*V >$$

B is injective, B, B^* are well-defined and continuous.

Well-Posedness

Theorem 1
The DPG variational formulation is well-posed with a mesh-independent inf-sup constant.

Theorem 2
There exists a mesh-independent $C > 0$:

$$\|u - u_{hp}\|_{L^2(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{hp}\|_{L^2(\Omega)}$$
$$+\|\hat{u} - \hat{u}_{hp}\|_{\tilde{H}^{1/2}(\Gamma_h^0)} + \|\hat{q} - \hat{q}_{hp}\|_{H^{-1/2}(\Gamma_h)}$$
$$\leq C \inf_{\boldsymbol{\sigma}_{hp}, u_{hp}, \hat{q}_{hp}, \hat{u}_{hp}} [...]$$

where $u_{hp}, \boldsymbol{\sigma}_{hp}, \hat{u}_{hp}, \hat{q}_{hp}$ is the DPG FE solution.

Optimal Test Norm

Define:

$$\|\boldsymbol{V}\|_o = \|B^* V\| = \sup_{\boldsymbol{U}} \frac{|b(\boldsymbol{U},\boldsymbol{V})|}{\|\boldsymbol{U}\|_U}$$

$$= \sup_{u,\sigma,\hat{u},\hat{q}} \frac{(u, -\mathsf{div}\boldsymbol{\tau})_\Omega + (\boldsymbol{\sigma}, \boldsymbol{\alpha}^{-1}\boldsymbol{\tau} - \boldsymbol{\nabla} v)_\Omega + <\hat{u}, \tau_n>_{\Gamma_h^0} + <v, \hat{q}>_{\Gamma_h}}{(\|u\|^2 + \|\boldsymbol{\sigma}\|^2 + \|\hat{u}\|^2 + \|\hat{q}\|^2)^{1/2}}$$

$$= \left(\|\mathsf{div}\boldsymbol{\tau}\|^2 + \|\boldsymbol{\alpha}^{-1}\boldsymbol{\tau} - \boldsymbol{\nabla} v\|^2 + \|[v]\|^2_{\Gamma_h^0} + \|[\tau_n]\|^2_{\Gamma_h} \right)^{1/2}$$

where

$$\|[v]\|_{\Gamma_h^0} = \sup_{w \in H(\mathsf{div},\Omega)} \frac{<v, w_n>_{\Gamma_h}}{\|\boldsymbol{w}\|_{H(\mathsf{div},\Omega)}}$$

$$\|[\tau_n]\|_{\Gamma_h} = \sup_{w \in H_0^1(\Omega)} \frac{<w, \tau_n>_{\Gamma_h^0}}{\|w\|_{H^1(\Omega)}}$$

Equivalence of Norms

We will show that the standard and optimal norms are equivalent, i.e.

$$\|\boldsymbol{V}\| \leq C\|\boldsymbol{V}\|_o \quad \text{and} \quad \|\boldsymbol{V}\|_o \leq C\|\boldsymbol{V}\|$$

The second inequality is straightforward, we will focus on the first one.
Conclusions:

- B^* is injective,
- b satisfies the inf-sup condition (B is bounded below).

Consequently, Nečas - Babuška (Generalized Lax-Milgram, Lions, Banach Closed Range) Theorem implies that the variational problem is well-posed. Theorem 2 follows.

Proof

Take $\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega_h), v \in H^1(\Omega_h)$. Denote

$$\begin{aligned} \boldsymbol{\alpha}^{-1}\boldsymbol{\tau} - \boldsymbol{\nabla} v =: &\quad \boldsymbol{f} \\ \mathrm{div}\boldsymbol{\tau} =: &\quad g \end{aligned}$$

Need to show the bounds:

$$\|\boldsymbol{\tau}\|_{H(\mathrm{div},\Omega_h)}, \|v\|_{H^1(\Omega_h)} \le C(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|[v]\|_{\Gamma_h^0} + \|[\tau_n]\|_{\Gamma_h})$$

Step 1: $\boldsymbol{f} = \boldsymbol{0}, g = 0$.
Consider the weighted Helmholtz decomposition:

$$\boldsymbol{\tau} = \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z}, \quad \psi \in H_0^1(\Omega), \boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$$

Potentials $\psi, \boldsymbol{\tau}$ are unique, orthogonal in the weighted $(\boldsymbol{\alpha}^{-1}\cdot, \cdot)$ L^2-product, and depend continuously upon $\boldsymbol{\tau}$.

Step 1: $f, g = 0$

$$\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h}$$

Step 1: $f, g = 0$

$$\begin{aligned}\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\ &= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla}v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h}\end{aligned}$$

Step 1: $f, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= -(\mathrm{div}\boldsymbol{\tau}, \psi)_{\Omega_h} + < \psi, \tau_n >_{\Gamma_h} + < v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0}
\end{aligned}
$$

Step 1: $f, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= -(\mathrm{div}\boldsymbol{\tau}, \psi)_{\Omega_h} + < \psi, \tau_n >_{\Gamma_h} + < v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0} \\
&= \frac{< \psi, \tau_n >_{\Gamma_h}}{\|\psi\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \frac{< v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0}}{\|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{H(\mathrm{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)}
\end{aligned}
$$

Step 1: $f, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= -(\mathrm{div}\boldsymbol{\tau}, \psi)_{\Omega_h} + < \psi, \tau_n >_{\Gamma_h} + < v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0} \\
&= \frac{< \psi, \tau_n >_{\Gamma_h}}{\|\psi\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \frac{< v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0}}{\|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{H(\mathrm{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)} \\
&\leq \sup_{w \in H_0^1(\Omega)} \frac{< w, \tau_n >_{\Gamma_h}}{\|w\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \sup_{w \in H(\mathrm{div},\Omega)} \frac{< v, w_n >_{\Gamma_h^0}}{\|\boldsymbol{w}\|_{H(\mathrm{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|
\end{aligned}
$$

Step 1: $f, g = 0$

$$
\begin{aligned}
\|\boldsymbol{\tau}\|^2_{\alpha^{-1}} &= (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (\boldsymbol{\alpha}^{-1}\boldsymbol{\tau}, \boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= (\boldsymbol{\tau}, \boldsymbol{\nabla}\psi)_{\Omega_h} + (\boldsymbol{\nabla} v, \boldsymbol{\nabla} \times \boldsymbol{z})_{\Omega_h} \\
&= -(\mathrm{div}\boldsymbol{\tau}, \psi)_{\Omega_h} + < \psi, \tau_n >_{\Gamma_h} + < v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0} \\
&= \frac{< \psi, \tau_n >_{\Gamma_h}}{\|\psi\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \frac{< v, (\boldsymbol{\nabla} \times \boldsymbol{z}) \cdot \boldsymbol{n} >_{\Gamma_h^0}}{\|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{H(\mathrm{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)} \\
&\leq \sup_{w \in H_0^1(\Omega)} \frac{< w, \tau_n >_{\Gamma_h}}{\|w\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \sup_{w \in H(\mathrm{div},\Omega)} \frac{< v, w_n >_{\Gamma_h^0}}{\|\boldsymbol{w}\|_{H(\mathrm{div},\Omega)}} \|\boldsymbol{\nabla} \times \boldsymbol{z}\|_{L^2(\Omega)} \\
&\leq C \left(\|[v]\|_{\Gamma_h^0} + \|[\tau_n]\|_{\Gamma_h} \right) \|\boldsymbol{\tau}\|_{\alpha^{-1}}
\end{aligned}
$$

Step 1: $f, g = 0$

Consequently,

$$\|\nabla v\|_{L^2(\Omega_h)} \leq C \left(\|[v]\|_{\Gamma_h^0} + \|[\tau_n]\|_{\Gamma_h} \right)$$

as well.

Discrete Poincaré Inequality:

$$\|v\|_{\Omega_h} \leq C \left(\|\nabla v\|_{\Omega_h} + \|[v]\|_{\Gamma_h^0} \right)$$

gives

$$\|v\|_{H^1(\Omega_h)} \leq C \left(\|[v]\|_{\Gamma_h^0} + \|[\tau_n]\|_{\Gamma_h} \right)$$

Step 2: $f, g \neq 0$

Let $\boldsymbol{\tau}_1 \in \boldsymbol{H}(\text{div}, \Omega), v_1 \in H_0^1(\Omega)$ such that

$$\begin{cases} \boldsymbol{\alpha}^{-1}\boldsymbol{\tau}_1 - \boldsymbol{\nabla} v_1 & = \boldsymbol{f} \\ \text{div}\boldsymbol{\tau}_1 & = g \end{cases}$$

Brezzi's Theory implies

$$\|\boldsymbol{\tau}_1\|_{H(\text{div},\Omega)}, |v_1\|_{H^1(\Omega)} \leq C(\|\boldsymbol{f}\| + \|g\|)$$

Final step: replace $\boldsymbol{\tau}, v$ with $\boldsymbol{\tau} - \boldsymbol{\tau}_1, v - v_1$ and use Step 1 result. Note that jump terms for $\boldsymbol{\tau} - \boldsymbol{\tau}_1, v - v_1$ are controlled by the original jump terms and norms of $\boldsymbol{\tau}_1, v_1$.

Generalization to Convection-Dominated Diffusion

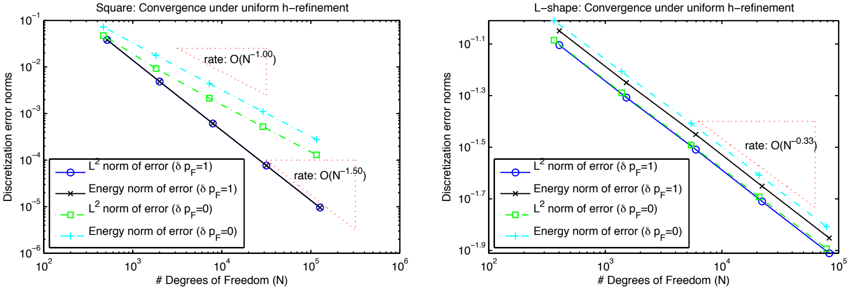
In Step 1, use the decomposition:

$$\boldsymbol{\tau} = (\boldsymbol{\alpha}\boldsymbol{\nabla}\psi + \boldsymbol{\beta}\psi) + \boldsymbol{\nabla} \times \boldsymbol{z}, \quad \psi \in H_0^1(\Omega), \boldsymbol{z} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$$

Numerical Experiments

Test problems:

- Square domain with $u(x, y) = \sin(\pi x)\sin(\pi y)$,
- L-shape domain with $u(r, \theta) = r^{2/3} \sin\left(\frac{2}{3}(\theta + \frac{\pi}{2})\right)$

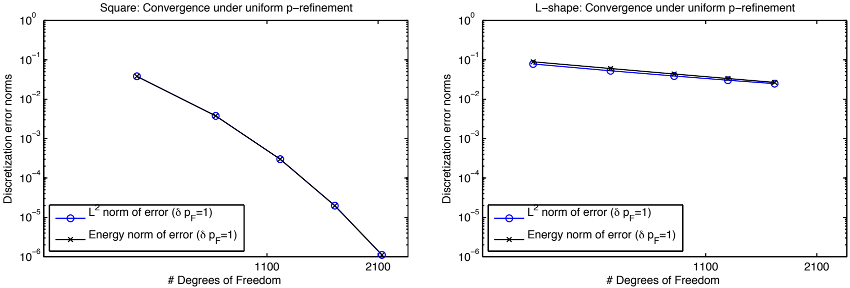


(a) The square case

(b) The case of the L-shaped domain

Figure: h-convergence rates for the two examples

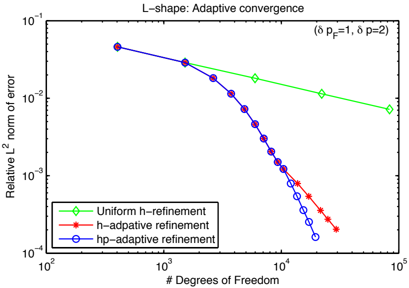
Uniform p-convergence rates



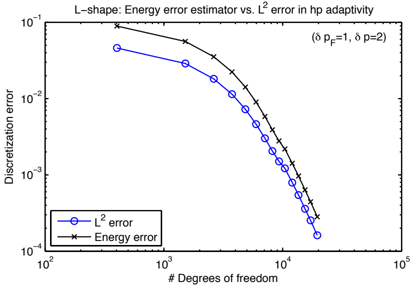
(a) Results from the square domain (b) Results from the L-shaped domain

Figure: p-convergence rates for the two examples

Adaptivity



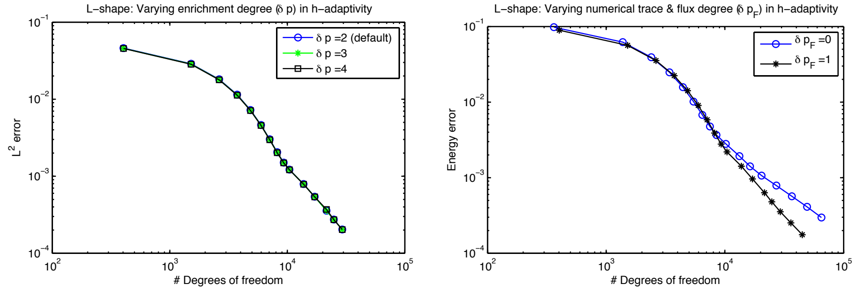
(a) Comparison of convergence of adaptive schemes



(b) Energy error estimator vs. L^2-error

Figure: Convergence curves from adaptive schemes

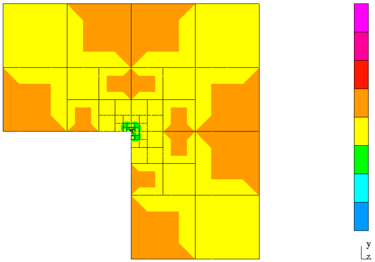
Adaptivity - cont.



(a) Effect of varying δp (b) Effect of varying δp_F

Figure: Convergence curves from adaptive schemes

Some Color to Finish



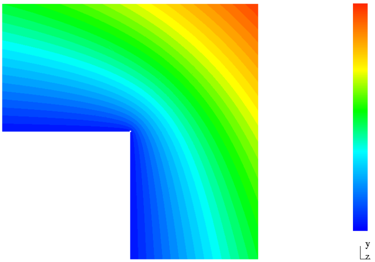


Figure: Left: The hp mesh found by the hp-adaptive algorithm after 15 refinements. (Color scale represents polynomial degrees.) Right: The corresponding solution u. (Color scale represent solution values.)

Thank You !

DPG references

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Happy Birthday Lars, Rick and Joe !!