1. Introduction.

The idea of conjugate or dual representations for finite-element approximations was introduced in [1] in connection with purely topological properties of finite element models; i.e., in the spirit of linear graph theory, if \( f(u) \) is a linear functional on a space \( M \) of finite element approximations \( u \), then there exists a consistent (conjugate) approximation of \( f \) which renders \( f(u) \) invariant under transformations from the unconnected model (i.e., the system of disjoint finite elements) to the global model (the connected elements). This idea led to the theory of conjugate approximation functions introduced in [2] and applied to various problems in elasticity theory in [3-6]. A detailed summary account, together with a few generalizations is given in [6].

In the present investigation, we expand the theory of conjugate approximations to include so-called mixed formulations of linear boundary-value problems. This amounts to a natural generalization of cases studied earlier in which the subspaces \( M_G = \text{span} \{ \phi_1(x), \ldots, \phi_G(x) \} \)

* University of Alabama.
generated by the usual displacement-type finite element formulations are self-dual. Also, all of the preliminary results in [5], which concerned situations in which different meshes are used to generate primal and dual approximations, are generalized and recast into a more meaningful framework. As a result, the concepts of "consistent" masses, forces, and even stresses [3,4] common to continuum mechanics applications can be expanded, and the notion of mixed and hybrid finite-element models (e.g. [7]) can be cast into a more general setting.

Following this introduction, we discuss several features of the theory of conjugate projections with special emphasis on discrete models of a linear operator $T$ and its adjoint $T^*$. We show that at least four sequences of projections of a class of boundary value problems involving $T^*T$ suggest themselves, and these lead directly to both the conventional and to mixed finite-element models. In Section 3 we discuss details of the finite-element formulation of mixed models, and in Section 4 we establish the relationship between our projection methods, the complementary variational principles of Noble and others [8-10], and the hypercircle method of Prager and Synge [11,12]. In terms of approximation of boundary-value problems, the notion of conjugate projections seems to suggest a broader collection of possibilities than either of the more conventional approaches. Finally, in Section 5 of the paper we address ourselves to certain preliminary questions of convergence and error estimation. There we establish convergence rates for our so-called primal, dual, and mixed finite-element models, the latter
being restricted to certain special choices of test functions for dual approximations.

2. Conjugate Projections and Approximations.

2.1 Conjugate subspaces. We record here a number of properties of dual and conjugate subspaces. First, we consider two linear vector spaces, \( U \) and \( V \), defined over the same field. For the moment, \( V \) is regarded as the dual of \( U \) and we denote by \( \langle u, v \rangle \) the bilinear scalar product \( s: U \otimes V \rightarrow \mathbb{F}^* \), which takes ordered pairs \( u, v (u \in U, v \in V) \) into scalars. Further, let \( \{ \phi_\alpha \} \) (\( \alpha = 1, 2, \ldots, G \)) denote a set of \( G \) linearly independent elements in \( U \) and \( \{ \chi_\alpha \} \) denote a set of \( G \) linearly independent elements in \( V \). The sets \( \{ \phi_\alpha \} \) and \( \{ \chi_\alpha \} \) define \( G \)-dimensional subspaces \( M_G \subseteq U \) and \( N_G \subseteq V \), respectively. If the \( G \times G \) matrix \( A \) defined by

\[
A^{\alpha \beta} = \langle \phi_\alpha, \chi_\beta \rangle
\]

(\( \alpha, \beta = 1, 2, \ldots, G \)) is nonsingular, then \( M_G \) and \( N_G \) are dual subspaces, and we use the notation \( N_G^* = M_G^* \). We also note that if \( A \) is nonsingular, we can construct sets of biorthogonal bases \( \{ \phi_\alpha \}, \{ \chi_\alpha \} \) by the transformations

\[
\chi_\alpha = \sum_{\lambda} A_\alpha^{-1, \lambda} \phi_\lambda ; \quad \phi_\alpha = \sum_{\lambda} b_\alpha^{\lambda, -1, \alpha} \chi_\lambda
\]

where \( A_\alpha^{-1, \lambda} \) are the elements of \( A^{-1} \). Clearly,
where $\delta^\beta_\alpha$ is the Kronecker delta and $\alpha, \beta = 1, 2, \ldots, G$.

Since transformations of the form (2.2) are always possible when $A$ is nonsingular, we choose either set as biorthogonal bases for $M_G$ and $M_G^*$.

By identifying bases $\{\phi_\alpha\}, \{\phi^\alpha\}$ with the property (2.3), we have effectively defined projection operators $\Pi: U \rightarrow M_G$ and $Q: V \rightarrow M_G^*$ in the following sense: if $u$ is an arbitrary element in $U$ and $v$ is an arbitrary element in $V$, the projection $\overline{u}$ of $u$ into $M_G$ and the projection $\overline{v}$ of $v$ into $M_G^*$ are of the form

\begin{equation}
\Pi u = \overline{u} = \sum_\alpha a^\alpha \phi_\alpha \quad ; \quad Qv = \overline{v} = \sum_\alpha a^\alpha \phi^\alpha
\end{equation}

where $a^\alpha = <u, \phi^\alpha>$ and $a^\alpha = <\phi^\alpha, v>$. We remark that without additional requirements, the set $\{\phi^\alpha\}$ with property (2.3) is, for a given $\{\phi_\alpha\}$, not unique.

Now suppose that $M_H$ is a subset of $M_G^*$ spanned by a set of $H$ elements $\omega^\Delta (\Delta = 1, 2, \ldots, H; H < G)$ given by

\begin{equation}
\omega^\Delta = \sum_\alpha R^\Delta_\alpha \phi^\alpha
\end{equation}

where $R^\Delta_\alpha$ are the entries in an $H \times G$ matrix. If we can find a set of elements $\{\omega^\Delta\} \subset M_G$ biorthogonal to $\omega^\Delta$, then there exists a $G \times H$ matrix $S$ such that
Clearly, $S$ is a left inverse of $R^T$ (i.e. $\sum_{\alpha} S^\alpha_{\Gamma} R^\alpha_{\Gamma} = \delta^\Gamma_{\Gamma}$; $\Delta, \Gamma = 1, \ldots, H$). If we denote by $\hat{u}$ and $\hat{v}$ the elements $\hat{u} = \sum_{\Delta} b^\Delta u^\Delta$ and $\hat{v} = \sum_{\Delta} b^{\Delta} v^\Delta$, where $b^{\Delta} = \langle u, v^\Delta \rangle$ and $b^\Delta = \langle \omega^\Delta, v \rangle$, then

\begin{equation}
(2.7)
\sum_{\Delta} b^\Delta a^\Delta = \sum_{\alpha} R^\alpha_{\alpha} a^\alpha \text{ and } b^\Delta = \sum_{\alpha} S^\alpha_{\Delta} a^\alpha
\end{equation}

where $a^\alpha$ and $a^\alpha$ are the coefficients in (2.4). Should it happen that the scalar product $\langle \hat{u}, \hat{v} \rangle$ remain invariant under the transformation (2.5), then we find that

\begin{equation}
(2.8)
\sum_{\Delta} b^\Delta b_{\Delta} = \sum_{\alpha} a^\alpha a^\alpha = \sum_{\alpha} \sum_{\alpha} R^\alpha_{\beta} S^\beta_{\alpha} a^\alpha a^\alpha
\end{equation}

This means that $\sum_{\alpha} R^\alpha_{\beta} S^\beta_{\alpha} = \delta^\beta_{\alpha}$; i.e. $R$ is a left inverse of $S$.

2.2 Hilbert spaces; projections of operators. Often we encounter cases in which $U$ and $V$ are Hilbert spaces and, consequently, self-dual. Suppose that such is the case, and that inner products in $U$ and $V$ are denoted $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot, \cdot]$, respectively. Then Gram matrices associated with the subspaces $M_G = \text{span} \{ \phi_{\alpha} \}_{\alpha=1}^G$ and $M_H = \text{span} \{ \omega^\Delta \}_{\Delta=1}^H$ can be constructed of the form
If \( \{ \phi_\alpha \} \) and \( \{ \omega^\Delta \} \) are linearly independent, \( G_{\alpha \beta} \) and \( H^{\Delta \Gamma} \) are nonsingular, and we can compute directly biorthogonal bases

\[
\phi^\alpha = \sum_\beta G_{\alpha \beta} \phi_\beta ; \quad \omega^\Delta = \sum_\Gamma H^{\Delta \Gamma} \omega^\Gamma
\]

where \( G_{\alpha \beta} \) and \( H^{\Delta \Gamma} \) are the inverses of \( G_{\alpha \beta} \) and \( H^{\Delta \Gamma} \).

Clearly, \( \{ \phi_\alpha, \phi^\beta \} = \delta_\alpha^{\beta} \) and \( \{ \omega^\Delta, \omega^\Gamma \} = \delta^\Delta_{\Gamma} \), so that the necessary ingredients for defining projections of any \( u \in U \) into \( M_G \) and any \( v \in V \) into \( N_H \) are available.

Clearly, if (2.5) holds, we could also compute the matrix

\[
H^{\alpha \beta} = [\phi^\alpha, \phi^\beta] ; \quad H^{\Delta \Gamma} = \sum_\alpha \sum_\beta R^\Delta_{\alpha \gamma} R_{\beta \gamma}
\]

The theory of conjugate projections described in [2.4] deals with such self-conjugate subspaces.

Now the biorthogonal bases in \( M_G \) and \( N_H \) are clearly independent of each other since there need now be nothing in common with the space \( U \) and \( V \). However, we open the door to many interesting applications if we assume that there exists a linear mapping \( T:U \rightarrow V \). If \( Q(T) \) is the range space of the operator \( T \), then the conjugate space \( Q(T)^* \subset V^* \) consists of those linear functionals of the form \( f(Tu) = [Tu, v] \), \( v \in V = V^* \). To be more specific, let \( U \) denote a Hilbert space consisting of functions \( u \) defined on a compact, convex subset \( R \) of \( E^n \) with a smooth boundary \( \partial R \), and let \( T \) denote a
bounded linear operator mapping $U$ into another Hilbert space $V$. If $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$ are, again, the inner products defined on $U$ and $V$, then we can identify an operator $T^* : V \to U$ as the formal adjoint of $T$ if it satisfies the relation

$$[v, Tu] = \{T^* v, u\} + [v, Bu]_{\partial R}$$

Here $B$ is a linear operator depending on $T$ which also has an adjoint $B^*$ in the sense that

$$[v, Bu]_{\partial R} = \{B^* v, u\}_{\partial R}$$

By $[\cdot, \cdot]_{\partial R}$ and $[\cdot, \cdot]_{\partial R}$ we mean, of course, inner products formed by restricting the domains of the entries to the boundary $\partial R$.

We recall that for any bounded linear operators $T: U \to V$, the null spaces $N(T)$ and $N(T^*)$ are closed subspaces of $U$ and $V$, respectively. If $N(T)_{\perp}$ and $N(T^*)_{\perp}$ denote orthogonal complements, then it can be shown that $U = N(T) + N(T)_{\perp}$ and $V = N(T^*) + N(T^*)_{\perp}$. Moreover (see [13]), $\overline{Q(T)} = N(T^*)_{\perp}$, $\overline{Q(T^*)} = N(T)_{\perp}$, $Q(T)_{\perp} = N(T^*)$, and $\overline{Q(TT^*)} = \overline{Q(T)}$ where $\overline{A}$ denotes closure of a set $A$ and $Q(T)$ is the range space of $T$. If $P$ is the orthogonal projection of $U$ onto $\overline{Q(T^*)}$, and $Q$ is the orthogonal projection of $V$ onto $\overline{Q(T)}$, then $Tu = TPu \forall u \in U$ and $T^* v = T^* Qv \forall v \in V$. A best approximate solution of $Tu = v$ is one which minimizes $\|w - v\|$ among all $w \in Q(T)$; such an element $w^*$ is such that $v - w^* \in Q(T)_{\perp} = Q(T^*)$ so that $T^* v = T^* w^*$. Con-
sequently, if $u$ is a best approximation of $Tu = v$, then $T^*u = T^*v = -f$ (say). Our method of conjugate projections consists of seeking such best approximations in finite-dimensional subspaces of $Q(T)$, $Q(T^*)$, etc., and is directly aimed at the problem of finding $u$ such that $T^*u + f = 0$ (see Art. 2.3).

If we consider cases in which $M_G \subset \mathcal{D}_T$ and $N_H \subset \mathcal{D}_{T^*}$, where $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ are the domains of $T$ and $T^*$, then a number of such approximations of $T$ and $T^*$ suggest themselves. Since, in general, $T(M_G)$ is not in $N_H$ and $T^*(N_H)$ is not in $M_G$, we can at least project these spaces into $N_H$ and $M_G$. This process leads to the following rectangular matrices associated with $T$ and $T^*$:

\[
P(T\phi_\alpha) = \sum_{\Delta} T_{\alpha\Delta} \omega_\Delta = \sum_{\Delta} T^{\alpha\Delta} \omega_\Delta
\]

(2.14)

$PT(M_G)$:

\[
P(T\phi_\alpha) = \sum_{\Delta} T_{\alpha\Delta} \omega_\Delta = \sum_{\Delta} T^{\alpha\Delta} \omega_\Delta
\]

(2.15)

$T^*(N_H)$:

\[
\Pi(T^*\phi_\alpha) = \sum_{\alpha} T^*_{\Delta\alpha} \phi_\alpha = \sum_{\alpha} T^{\Delta\alpha} \phi_\alpha
\]

Here $P: \mathcal{V} \rightarrow N_H$ and $\Pi: \mathcal{U} \rightarrow M_G$ are projection operators.
GENERALIZED CONJUGATE FUNCTIONS

defined by the bases $\phi_\alpha$ and $\omega_\Delta$; i.e.

$$\Pi u = \bar{u} = \sum_{\alpha} \{u, \phi_\alpha\} \phi_\alpha = \sum_{\alpha} \{u, \phi_\alpha\} \phi_\alpha$$

(2.16)

$$P v = \bar{v} = \sum_{\Delta} [y, \omega_\Delta] \omega_\Delta = \sum_{\Delta} [y, \omega_\Delta] \omega_\Delta$$

and

$$T_{\Delta\alpha} = [\omega_\Delta, T_{\phi_\alpha}] ; \quad T_{\alpha\Delta} = [\omega_\Delta, T_{\phi_\alpha}]$$

(2.17)

$$T_{\alpha\alpha} = [\omega_\Delta, T_{\phi_\alpha}] ; \quad T_{\alpha\alpha} = [\omega_\Delta, T_{\phi_\alpha}]$$

and

$$T_{\Delta\alpha}^* = \{T_{\omega_\Delta}, \phi_\alpha\} ; \quad T_{\alpha\Delta}^* = \{T_{\omega_\Delta}, \phi_\alpha\}$$

(2.18)

$$T_{\alpha\alpha}^* = \{T_{\omega_\Delta}, \phi_\alpha\} ; \quad T_{\alpha\alpha}^* = \{T_{\omega_\Delta}, \phi_\alpha\}$$

If we denote

$$B_{\Delta\alpha}^\alpha = \{B^*, \omega_\Delta, \phi_\alpha\}_{\beta R}, \quad B_{\alpha\Delta}^\alpha = \{B^*, \omega_\Delta, \phi_\alpha\}_{\beta R}$$

(2.19)

$$B_{\alpha\alpha}^\alpha = \{B^*, \omega_\Delta, \phi_\alpha\}_{\beta R}, \quad B_{\alpha\alpha}^\alpha = \{B^*, \omega_\Delta, \phi_\alpha\}_{\beta R}$$

then, in view of (2.12), the matrices in (2.17) are related to those in (2.18) according to

$$T_{\Delta\alpha} = T_{\alpha\Delta}^* + B_{\Delta\alpha} ; \quad T_{\alpha\Delta} = T_{\Delta\alpha}^* + B_{\alpha\Delta}$$

(2.20)

$$T_{\alpha\alpha} = T_{\alpha\alpha}^* + B_{\alpha\alpha} ; \quad T_{\alpha\alpha} = T_{\alpha\alpha}^* + B_{\alpha\alpha}$$

Obviously, if $B_{\omega_\Delta}^\alpha$ is zero, then the $B_{\alpha\alpha}^\alpha$ matrices vanish.
and each matrix $T^*$ of (2.18) is merely the transpose of the corresponding matrix in (2.17). Moreover, if (2.10) holds, we have

$$T_{\Delta \alpha} = \sum_{\beta} T'_{\Delta \beta} G_{\beta \alpha} = \sum_{\Gamma} H_{\Delta \Gamma} T'_{\Gamma \alpha} = \sum_{\beta} \sum_{\Gamma} H_{\Delta \Gamma} G_{\alpha \beta} T'_{\beta}$$

(2.21)

$$T^*_{\Delta \alpha} = \sum_{\Gamma} H_{\Delta \Gamma} T^*_{\Gamma \alpha} = \sum_{\beta} T^*_{\Delta \beta} G_{\alpha \beta} = \sum_{\beta} \sum_{\Gamma} H_{\Delta \Gamma} G_{\alpha \beta} T^*_{\beta}$$

etc. Similar transformations hold for the matrices $B$ of (2.19).

2.3 Approximation by conjugate projections. We now address the question of determining approximate solutions of mixed boundary-value problems of the type

(2.22) \[ T^* T u + f = 0 \text{ in } \mathcal{R} \]

\[ B(g - u) = 0 \text{ on } \partial \mathcal{R}_1 ; \quad B^*(Tu) - S = 0 \text{ on } \partial \mathcal{R}_2 \]

Here $u, f \in \mathcal{U}$, $T$ is a linear operator from $\mathcal{U}$ into $\mathcal{Y}$ and $T^*$ is its formal adjoint in the sense of (2.12), $B g$ and $S$ are prescribed functions on portions $\partial \mathcal{R}_1$ and $\partial \mathcal{R}_2$ of the boundary $\partial \mathcal{R} = \partial \mathcal{R}_1 \oplus \partial \mathcal{R}_2$, and $B$ and $B^*$ satisfy (2.13). As noted earlier, the problem (2.22) can be split into the so-called canonical form by setting

(2.23) \[ Tu = \nu \text{ in } \mathcal{R} ; \quad B(g - u) = 0 \text{ on } \partial \mathcal{R}_1 \]

(2.24) \[ T^* \nu = -f \text{ in } \mathcal{R} ; \quad B^* \nu - S = 0 \text{ on } \partial \mathcal{R}_2 \]

Our method of approximation consists of seeking
weak solutions of (2.23) or (2.24) and (2.25) in subspaces \( M_G \), \( M_G \times N_H \), and \( N_H \) by use of conjugate projections. A number of possibilities suggest themselves:

1. **The primal projection.** This amounts to the conventional Ritz-Galerkin approximation of (2.22). We seek \( \Pi u \) instead of \( u \) and require that the projection of the residual into \( M_G \) vanish; i.e.

\[
\Pi (T^*T(\Pi u) + f) = 0 \quad \text{in} \quad \mathbb{R} \\
\Pi (B^*(T(\Pi u) - S) = 0 \quad \text{on} \quad \partial \mathbb{R}_2
\]

(2.25)

We may also require that

\[
(B(g - \Pi u)) = 0 \quad \text{on} \quad \partial \mathbb{R}_1
\]

(2.26)

However, this requirement can be intrinsically satisfied if the basis functions \( \phi_\alpha \) are suitably chosen.

Since the solution \( u^* \) is unknown, we can only assert that its projection into \( M_G \) will be of the form \( \Pi u^* = \sum a^\alpha \phi_\alpha \) (see (2.16)). Substituting this into (2.25), we then select the coefficients \( a^\alpha \) so as to reduce the residual to zero; i.e.

\[
\sum_{\beta} \{ T^*T \left( \sum_{\alpha} a^\alpha \phi_\alpha \right), \phi_\beta \} + \sum_{\beta} \{ f, \phi_\beta \} \phi_\beta^2 = 0
\]

\[
\sum_{\beta} \{ B^*T \left( \sum_{\alpha} a^\alpha \phi_\alpha \right) - S, \phi_\beta \} \phi_\beta^2
\]

Since the \( \phi_\beta \) are linearly independent, these relations lead, with the aid of (2.12), to the system of algebraic equations.
(2.27) \[ \sum_{\beta} K_{\alpha \beta} a^\beta + f_\alpha = 0 \]

where
\[ K_{\alpha \beta} = \{ T \phi_\alpha, T \phi_\beta \} - \{ B^* T \phi_\alpha, \phi_\beta \} \in \mathcal{R}_1 \]

(2.28) \[ f_\alpha = \{ f, \phi_\alpha \} - \{ S, \phi_\alpha \} \in \mathcal{R}_2 \]

Solving (2.27) for \( a^\alpha \) determines, of course the approximate solution \( \sum_{\alpha} a^{\alpha} \phi_\alpha \).

2. The primal-dual projection. We now turn to the canonical forms (2.23) and (2.24) and assume that \( T^* y \in Q(\Pi) \). We seek, instead of the solution of \( y^* \) of (2.24) and element \( \bar{y} = \sum b_{\alpha} \omega_{\alpha} \in \mathcal{N}_H \) such that

(2.29) \[ \Pi(T^* P(\bar{y}) + f) = 0 \text{ in } \mathcal{R}_1 \quad \Pi(B^* P(\bar{y}) - S) = 0 \text{ on } \partial \mathcal{R}_2 \]

where \( P \) is the projection operator described in (2.16). Observing that
\[ \{ T^* P(\bar{y}), \phi_\alpha \} = \left[ \sum_{\Delta} b_{\Delta} \omega_{\Delta}, T \phi_\alpha \right] - \{ S, \phi_\alpha \} \in \mathcal{R}_2 \quad \{ B^* P(\bar{y}), \phi_\alpha \} \in \mathcal{R}_1 \]

we obtain from (2.29) the following system of equations:

(2.30) \[ \sum_{\Delta} (T_{\alpha}^\Delta - B_{\alpha}^{* \Delta 1}) b_{\Delta} + f_\alpha = 0 \]

Here \( T_{\alpha}^\Delta \) is the array defined in (2.17), \( B_{\alpha}^{* \Delta 1} = \{ B^* \omega_{\Delta}, \phi_\alpha \} \in \mathcal{R}_1 \), and \( f_\alpha \) is given by (2.28). Since \( M_G \) and
GENERALIZED CONJUGATE FUNCTIONS

$N_H$ are, in general, of different dimensions, $T_{\alpha}^\Delta$ is rectangular and no unique solution of (2.30) exists.

3. The dual-primal projection. Now we assume that $Tu \in Q(P)$ and we require that the projection of the residual of (2.23) vanish in $N_H$:

$$P(T\nu - Py) = 0 \text{ in } K; \quad P(B(g - Nu)) = 0 \text{ on } \partial K.$$ (2.31)

This leads to the system

$$
\sum_{\alpha} a^\alpha T_{\alpha}^\Delta - \sum_{\Gamma} H_{\Gamma}^{\Delta} b_{\Gamma} = 0
$$ (2.32)

where $T_{\alpha}^\Delta$ is defined by (2.17) and $H_{\Gamma}^{\Delta}$ is given by (2.9).

While $H_{\Gamma}^{\Delta}$ is nonsingular, insufficient information is provided by (2.32) to determine both $a^\alpha$ and $b_{\Delta}$. However, use of (2.30) together with (2.32) yields enough information to determine both $a^\alpha$ and $b_{\Delta}$, and the resulting system is said to describe a mixed discrete model of the problem (2.22). Moreover, (2.32) can be used to construct an alternate (but not equivalent) form of (2.27). Observing that

$$b_{\Gamma} = \sum_{\Delta} H_{\Gamma\Delta} T_{\alpha}^\Delta a^\alpha$$ (2.33)

where $H_{\Gamma\Delta}$ is the inverse of $H_{\Gamma\Delta}^{\Delta}$, we obtain from (2.30)

$$\sum_{\beta} K_{\alpha\beta} a^\beta + f_{\alpha} = 0$$ (2.34)
Here if $\mathbf{M}^\Delta = (T^\Delta - B^\Delta_1)$, then

$$\tilde{\mathbf{K}}_{\alpha\beta} = \sum_{\Delta} \sum_{\Gamma} \mathbf{M}^\Delta_{\alpha \delta} H_{\delta \Gamma} T^\Delta_{\delta \beta}$$

A comparison of (2.34) and (2.35) with (2.27) and (2.28) illustrates clearly the difference between the fundamental matrices $\mathbf{K}_{\alpha\beta}$ and $\tilde{\mathbf{K}}_{\alpha\beta}$ obtained via the direct (primal) and the mixed formulations.

4. The dual projection. In this case, $u$ is regarded as prescribed, $Tu \in \mathcal{V}$, and $\mathcal{N}_H$ is a subspace of the set of elements $v$ such that $T^* v = 0$ in $\mathcal{R}$ and $B^* v = 0$ on $\partial \mathcal{R}_2$. We consider the vanishing of the projection of the residual $Tu - P(v)$ in $\mathcal{N}_H$:

$$P(Tu - P(v)) = 0$$

Noting that

$$P^2(y) = P(y) = \sum_{\Delta} b_{\Delta} \omega^\Delta$$

and

$$[Tu, \omega^\Delta] = \{B^* \omega^\Delta, g\}_{R_1}$$

we obtain from (2.36) the system of equations

$$- \sum_{\Gamma} H_{\Gamma \beta} b_{\Gamma} + \eta_{\Delta} = 0$$

wherein

$$\eta_{\Delta} = \{B^* \omega^\Delta, g\}_{R_1}$$
5. Discrete Mappings. A number of other models can be obtained by using the idea of projections. For example, consider the model obtained by successive projections of each image of the decomposed operator $T^*T$ into an appropriately defined subspace. By this we mean the following:

$$\Pi \left( T^* \left( PT(\Pi(u)) \right) \right) + \Pi f = 0$$

i.e.

$$\sum_{\alpha} \sum_{\Delta} \sum_{\beta} \left[ \{ T^* \{ T_{\alpha}^{\beta} \phi_{\beta} \}, \phi_{\alpha} \} \phi_{\alpha} + f_{\alpha} \phi_{\alpha} \right] = 0$$

This yields a discrete model similar in form to (2.22):

$$\sum_{\beta} \sum_{\Delta} T_{\alpha}^{\Delta} T_{\beta}^{\Delta} \phi_{\beta} + f_{\alpha} = 0$$

Alternately, use of (2.20) leads to the equivalent form

$$\sum_{\beta} \sum_{\Delta} (T_{\alpha}^{\Delta} + B_{\alpha}^{\Delta}) T_{\beta}^{\Delta} \phi_{\beta} + f_{\alpha} = 0$$

Still other forms could be cited.

3. Finite element approximations

3.1 The finite element method. We view the finite element method as both a method of interpolation and as a method for the systematic construction of basis functions for Ritz-Galerkin approximations on irregular domains. We shall outline briefly the essential features of the method following the ideas in [6].

Consider an open, bounded region $\tilde{R}$ in $n$-dimensional euclidean space $\mathbb{E}^n$, with closure $\bar{R} = R \cup \partial R$. 
and let $x = (x_1, x_2, \ldots, x_n)$ denote a point in $\bar{R}$. A finite element model $\bar{R}$ of $\bar{R}$ is the union of a finite number $E$ of closed bounded subregions $\bar{R}_e$ of $E^n$, $\bar{R}_e$ being the closure of an open region $\mathcal{R} \subset E^n (\bar{R}_e = \mathcal{R} + \partial \mathcal{R}_e; e = 1, 2, \ldots, E)$. The subregions $\bar{R}_e$ are called finite elements, $\mathcal{R}$ is referred to as the connected model of $\bar{R}$, and the open elements $\mathcal{R}_e$ are disjoint in $E^n$:

$$\mathcal{R} = \bigcup_{e=1}^{E} \mathcal{R}_e; \quad \mathcal{R}_e \cap \mathcal{R}_f = \emptyset, \ e \neq f$$

In the connected model $\mathcal{R}$, we identify a finite number $G$ of points, called global nodes, and we label these consecutively $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^G$. Likewise, we identify within each finite element $\mathcal{R}_e$ a number $N_e$ of points, called local nodes, and we label them consecutively $x_e^1, x_e^2, \ldots, x_e^{N_e}; e = 1, 2, \ldots, E$. Using the notation developed in [6], we refer to a typical global node as $x^\alpha (\alpha = 1, 2, \ldots, G)$ and a typical local node as $x_e^N (N = 1, 2, \ldots, N_e; e = 1, 2, \ldots, E)$. We assume, of course, that the proper correspondence between points in each $\mathcal{R}_e$ and points in $\mathcal{R}$ exists, as well as a proper correspondence between nodal points $x_e^N$ in $\mathcal{R}_e$ and nodal points $x^\alpha$ in $\mathcal{R}$, so that the elements fit smoothly together to form $\bar{R}$ (see the compatibility requirements, [6, p. 36]). We also remark that in some instances it is convenient to use different local coordinate systems for each finite element and a different coordinate system for the connected model (see, for example, [6, p. 44]); however, for simplicity we assume here that all local and global frames are coincident.

Conceptually, either a given $\mathcal{R}$ is decomposed
into distinct finite elements $\overline{R}_e$, $e = 1, 2, \ldots, E$, or, as is usually done in computations, distinct elements are connected appropriately together to form $\overline{R}$. Mathematically, the connectivity and decomposition of a finite element model are established by the Boolean mappings

$$x^\alpha = \sum_{N=1}^{e} \Lambda_{N} x^\alpha_{N} (e \text{ fixed}); \quad x^N_e = \sum_{\alpha=1}^{G} \Omega_{\alpha} x^\alpha_e$$

Here $\alpha = 1, 2, \ldots, G$, $N = 1, 2, \ldots, N_e$, $e = 1, 2, \ldots, E$,

$$\Omega_{\alpha} = \begin{cases} 1 & \text{if node } x^\alpha_e \text{ of } \overline{R} \text{ is coincident with node } x^N_{-e} \text{ of } \overline{R}_e \\ 0 & \text{if otherwise,} \end{cases}$$

$$\Lambda_{N} = \begin{cases} (e)^N_{\alpha} & \text{if node } x^\alpha_e \text{ of } \overline{R} \text{ is coincident with node } x^N_{-e} \text{ of } \overline{R}_e \\ A \text{ and } \Omega \text{ have the property} & \sum_{\alpha=1}^{G} \Omega_{\alpha} A_{M} = \delta_{M} \quad (e \text{ fixed}); \quad \sum_{N=1}^{N_e} \Lambda_{N} \Omega_{\alpha} = \begin{cases} \delta_{\alpha}^\beta x^\alpha, x^\beta & \in \overline{R}_e \\ 0 & \in \overline{R}_e \end{cases}$$

The symbolism (3.2) effectively represents a renumbering of local and global node labels.

Now consider a function $U(x) \in C^m(\overline{R})$. The restriction $u_e(x)$ to $\overline{R}_e \subset \overline{R}$ is the function

$$u_e(x) = \begin{cases} U(x) & \text{if } x \in \overline{R}_e \\ 0 & \text{if } x \notin \overline{R}_e \end{cases}$$
In the finite-element method, a local representation of $u_e(x)$ of order $q$ is defined as the function

\begin{equation}
  u_e(x) = \sum_{N=1}^{\infty} \sum_{|\alpha| \leq q} a_N^\alpha \psi_N^\alpha(x)
\end{equation}

wherein $a_N^\alpha$ are constants, $\alpha$ is a multi-integer (i.e., $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+^n$ is the set of $n$-tuples of nonnegative integers), $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$; $\psi_N^\alpha(x)$ are the local interpolation functions of order $q$, and are defined so as to have the properties

\begin{equation}
  \psi_N^\alpha(x) = 0 \text{ if } x \notin R_e
\end{equation}

\begin{equation}
  D_{\beta}^M \psi_N^\alpha(x_e) = \delta_N^M \delta_{\beta_1}^1 \delta_{\beta_2}^2 \ldots \delta_{\beta_n}^n
\end{equation}

Here $\delta_N^M$, $\delta_{\beta_1}^1$, $\ldots$, $\delta_{\beta_n}^n$ are Kronecker deltas and we have again used the multi-integer notation,

\begin{equation}
  D_\beta u \equiv \beta \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_n^{\beta_n}}
\end{equation}

Therefore, in view of (3.7),

\begin{equation}
  D_\beta^M u_e(x_e) = a_N^\alpha \beta \; ; \; |\beta| \leq q
\end{equation}

The global finite-element representation of $U(x)$ is the function

\begin{equation}
  \overline{U}(x) = \sum_{\lambda=1}^{G} \sum_{|\gamma| \leq q} A_\lambda^\gamma \phi_\lambda^\gamma(x)
\end{equation}
where

$$A^\lambda_{\alpha} = \sum_{N=1}^{\mathcal{G}} \Omega_{N(e)}^\alpha N^\alpha_{\lambda} (e \text{ fixed}) ;$$

(3.10)

$$N(e) = \sum_{\lambda=1}^{\mathcal{G}} \Omega^\lambda N^\lambda_{\alpha} ,$$

and $\phi^\alpha_{\lambda}(x)$ are the \textit{global} interpolation functions

(3.11)

$$\phi^\alpha_{\lambda}(x) = \bigcup_{e=1}^{\mathcal{E}} \left\{ \sum_{N=1}^{\mathcal{G}} \Omega^\lambda N^\lambda_{\alpha} (x) \right\}$$

It is easily verified that

(3.12)

$$D_\beta \phi^\alpha_{\lambda}(x^\mu) = \delta^\alpha_{\beta} \delta^\mu_{\lambda}$$

so that

(3.13)

$$D_\beta \overline{u}(x^\mu) = A^\mu_{\beta},$$

$$|\beta| \leq q ; \quad \lambda, \mu = 1,2, \ldots, G, \quad \alpha, \beta \in \mathbb{Z}_+^n$$

Hence, if we set

(3.14)

$$a^N_{\beta} = D_\beta u(x^N) \quad \text{and} \quad A^\mu_{\beta} = D_\beta \overline{u}(x^\mu)$$

\textbf{3.2 Finite Element Approximations by Conjugate Projections.}

The beauty of the finite element method is that the essential features of an approximation can be described locally for typical disconnected elements. Corresponding global
approximations are then obtained routinely by means of the transformations (3.2) and (3.10).

1. Primal finite-element model. The direct finite-element approximation of (2.22) leads to a primal model of the form (2.30). Consider $M_G$ to be a subspace of $U$ spanned by the functions $\varphi(x)$ of (3.11). Then we seek coefficients $A^\lambda$ among projections of the form

$$ (3.15) \quad U(x) = \sum_\lambda A^\lambda \varphi(x) = E U \sum_\lambda \sum_N (e)N A^\lambda \psi_N(x) $$

which satisfy

$$ (3.16) \quad \sum_\beta K_{\alpha \beta} A^\beta + f_\alpha = 0 $$

Here, and in the following discussion, we have omitted the multi-indices for simplicity; $K_{\alpha \beta}$ and $f_\alpha$ are defined by (2.28) and, due to properties (3.12) of $\varphi(x)$, the coefficients $A^\lambda$ now can be interpreted as actual values of $D_\alpha U(x)$ at nodal points $x^\beta$.

Due to the particular choice of basis functions, we observe that

$$ K_{\alpha \beta} = [T \lambda \alpha, T \lambda \beta] - \{B^* \lambda \alpha, \lambda \beta \} \delta_{\alpha \beta} = $$

$$ E U \sum_{e=1}^E \sum_{f=1}^F \sum_N (e)N (f)M \{T \psi_N(e), T \psi_M(f) \} - $$

$$ \{B^* \psi_N(e), \psi_M(f) \} \delta_{\alpha \beta} $$

and

648
GENERALIZED CONJUGATE FUNCTIONS

\[ f_{\alpha} = \{ f, \phi_{\alpha} \} = \bigcup_{e=1}^{E} \sum_{\alpha}^{(e)} \bigcup_{\Omega}^{N} \{ f, \psi_{N}^{(e)} \} \]

However, \([ T_{\Psi_{N}^{(e)}}, T_{\Psi_{M}^{(f)}} ] = 0\) and \(\{ B^{*} T_{\Psi_{N}^{(e)}}, \psi_{M}^{(f)} \}_{\Omega} = 0\)
\(\text{for } e \neq f\). Therefore

\[ K_{\alpha \beta} = \sum_{e} \sum_{N} \sum_{M}^{(e)} \Omega^{N} \{ k_{N}^{(e)}(e)^{M} \}_{\Omega}^{\alpha \beta} \]

(3.17)

\[ f_{\alpha} = \sum_{e} \sum_{N}^{(e)} \Omega^{N} f_{N}^{(e)} \]

where \(k^{(e)}\) and \(f_{N}^{(e)}\) are the localizations of \(K_{\alpha \beta}\) and \(f^{(e)}\):

\[ k_{NM}^{(e)} = [ T_{\Psi_{N}^{(e)}}, T_{\Psi_{M}^{(f)}} ] - \{ B^{*} T_{\Psi_{N}^{(e)}}, \psi_{M}^{(f)} \}_{\Omega} = 0 \]

(3.18)

\[ f_{N}^{(e)} = \{ f, \psi_{N}^{(e)} \} - \{ S, \psi_{N}^{(e)} \}_{\Omega} R_{2} \]

In view of the fact that \(\sum_{\lambda}^{(e)} N_{\lambda} A_{\lambda}^{N} = a_{N}^{(e)}\), we can see that (3.16) can be written

\[ \sum_{e} \sum_{N}^{(e)} \Omega^{N} \{ \sum_{M}^{(e)} k_{NM}^{(e)} a_{M}^{(e)} + f_{N}^{(e)} \} = 0 \]

(3.19)

which implies that locally

\[ \sum_{M}^{(e)} k_{NM}^{(e)} a_{M}^{(e)} + f_{N}^{(e)} = 0 \]

(3.20)

for each \(e = 1, 2, \ldots, E\).
Equation (3.20) is the local finite-element model of (2.1). Again, (3.20) can be generated locally and the global model (3.16) can be obtained directly by using the transformations (3.17).

2.3. Primal-dual and dual-primal (mixed) finite-element models. Since both of the approximations (2.30) and (2.32), must, in general, be used jointly to determine a unique projection into $M_G$ and $N_H$, we must, in general, develop local finite element approximations of these equations jointly. Then considering the restriction $v_e(x)$ of a model $v(x)$ of $y \in \mathcal{V}$ to a typical finite element $e$ to be of the form

\begin{equation}
(3.21) \quad v_e(x) = \sum_{i=1}^{R} b^{(e)}_i \psi^i(e)(x)
\end{equation}

where $\psi^i(e)(x)$ are linearly independent functions with local support (i.e., $\psi^i(e)(x) = 0$ if $x \notin \overline{R_e}$) and $R < \infty$. Following the same procedure used earlier, now on a local level, yields as the local counterparts of (2.30) and (2.32),

\begin{equation}
(3.22) \quad \sum_{i=1}^{R} t^{(e)}_i b^{(e)}_i + f^{(e)}_N = 0 ;
\end{equation}

\begin{equation}
\sum_{N}^{N_e} t^{(e)}_i a^{(e)}_N - \sum_{j}^{R} F^{(e)}_i b_j = 0
\end{equation}

Here $a^{(e)}_N$ are the coefficients of the local approximation.
GENERALIZED CONJUGATE FUNCTIONS

\[ u_e(x) = \sum_{N} a_N^{(e)} \psi_N^{(e)}(x) \], \quad f_N^{(e)} \text{ are defined in (3.18), for}

convenience, \( \forall k^{(e)}_1 = 0 \), and

\[ (3.23) \quad t_N^{(e)i} = [T\psi_N^{(e)}, \psi_i^{(e)}] \] ; \( F_{ij}^{(e)} = [\psi_i^{(e)}, \psi_j^{(e)}] \)

Solving (3.22) for \( b_j^{(e)} \) and reorganizing terms, we arrive at the basic equation analogous to (2.34)

\[ (3.24) \quad \sum_{M} k_{NM}^{(e)} a_M^{(e)} + f_N^{(e)} = 0 \]

where

\[ (3.25) \quad k_{NM}^{(e)} = \sum_{i,j} t_N^{(e)i} F_{ij}^{(e)} \]

Globally, we then obtain (2.35) with

\[ (3.26) \quad \tilde{K}_{\alpha\beta} = \sum_{e} \sum_{N,M} \Omega_{N}^{(e)} k_{NM}^{(e)} M \Omega_{M}^{(e)} \]

\( \alpha, \beta = 1, 2, \ldots, G \) and \( f_\alpha \) given by (3.17).

A comparison of (3.26) with (2.34) leads to some interesting transformations. Suppose that there exist, associated with each finite element, \( R_e \times H \) matrices \( \mathcal{R}_e \) with the property that

\[ (3.27) \quad \mathcal{T}_{\alpha}^{\Delta} = \sum_{e} \sum_{N} \sum_{i} R_e^{(e)} \Omega_{N}^{(e)} t_N^{(e)i} \mathcal{R}_i^{\Delta} \]

\( \alpha = 1, 2, \ldots, G \); \( \Delta = 1, 2, \ldots, H \), where \( \mathcal{T}_{\alpha}^{\Delta} \) is defined in
(2.17). Since $\phi_\alpha(x)$ is given by (3.11), we obtain the following relations:

\[
\omega^\Delta(x) = \sum_e \sum_i \sum_{\Delta} R_i^e \Delta \psi_i^e(x);
\]

(3.28)

\[
b_i^e = \sum_{\Delta} b_\Delta R_i^e \Delta
\]

(3.28)

\[
\Delta F = \sum_e \sum_{i,j} R_i^e \Delta i_j^e R_j^e F_i^e R_i^e
\]

4. Dual finite-element model. Locally, we have

\[
\sum_{i,j} F_{ij}^e b_i^e + g_i^e = 0
\]

(3.29)

where $g_i^e = \{B_i^e, \psi_i^e\}_{\partial R_e}$ (g is regarded as prescribed on all of $\partial R_e$) and $F_{ij}^e$ is defined in (3.23). To provide a means for establishing the connectivity of the model, we introduce a local transformation of $b_i^e$ to nodal quantities $\omega_N^e$ according to

\[
\omega_N^e = \{B^e, \psi_N^e\}_{\partial R_e} = \sum_{i} s_i^e b_i^e
\]

(3.30)

wherein

\[
s_1^e = \{B^e, \psi_1^e\}_{\partial R_e}
\]

(3.31)
Introducing nodal measures \( w_{(e)}^{N} \) such that \( \sum_{N} w_{(e)}^{N} = \sum_{i} b_{i} g_{i}^{i} \), we find that

\[
(3.32) \quad g_{(e)N}^{i} = \sum_{N} s_{(e)N}^{i} w_{(e)}^{N}
\]

Thus we have locally

\[
(3.33) \quad \sum_{M} k_{(e)NM}^{i} w_{(e)}^{M} = q_{(e)N}^{i}
\]

and globally

\[
(3.34) \quad \sum_{\beta} k_{\alpha\beta}^{i} w_{\beta}^{i} = q_{\alpha}
\]

where

\[
\hat{k}_{(e)NM}^{i} = \sum_{i,j} s_{(e)N}^{i} F_{ij}^{e} s_{(e)M}^{j}
\]

\[
\hat{k}_{\alpha\beta}^{i} = \sum_{e} \sum_{N,M} k_{(e)NM}^{i} \Omega_{\alpha\beta}^{N}(e) \Omega_{\alpha\beta}^{NM}(e)
\]

\[
(3.35) \quad q_{\alpha} = \sum_{e} \sum_{N} \Omega_{\alpha\beta}^{N}(e) q_{(e)N}^{i}
\]

\[
(3.35) \quad w_{(e)}^{N} = \sum_{\alpha} \Omega_{\alpha\beta}^{N}(e) w_{\beta}^{i}
\]

4.1 Complementary variational principles. The theory of $T^*T$ operators, initiated by Von Neumann [14] in 1932 and extended by Kato [15], Fujita [16], and others, has been revived in recent times in the study of complementary variational principles by Noble [8], Arthurs [9], and Robinson [10]. We shall summarize the essential features of this theory and show its relation to the notions of projections developed previously.

Following Noble and others (e.g. [8-10]), we begin by considering the bilinear functional

$$J(u, v) = [v, T^*T] - \frac{1}{2}[v, v] + \{f, u\} +$$
$$+ \{B^*v, g-u\}_{\partial \Omega_1} - \{S, u\}_{\partial \Omega_2}$$

(4.1)

$$= \{T^*v, u\} - \frac{1}{2}[v, v] + \{f, u\} +$$
$$+ \{B^*v - S, u\}_{\partial \Omega_2} + \{B^*v, g\}_{\partial \Omega_1}$$

If it is assumed that the conditions (2.23) are identically satisfied (i.e., $T^*T = v$ in $\Omega$ and $B(g - u) = 0$ on $\partial \Omega_1$), then we can obtain from $J(u, v)$ the new functional

$$I(u) = \frac{1}{2} [T^*T, u] + \{f, u\} - \{S, u\}_{\partial \Omega_2} +$$
$$+ \{B^*T^*, g - u\}_{\partial \Omega_1}$$

(4.2)
We note that the last term in this functional vanishes in the case \( B = 1 \); i.e., when \( u \) takes on the prescribed variation \( g \) on \( \partial R_1 \).

Likewise, if it is assumed that the conditions (2.24) are identically met (i.e., if \( T^* v + f = 0 \) in \( R \) and \( B^* v - S = 0 \) on \( \partial R_2 \) ), then we obtain from \( J(u,v) \) the new functional

\[
(4.3) \quad K(v) = -\frac{1}{2} \{ v, v \} + \{ B^* v, u \} \quad \text{on} \quad \partial R_1
\]

Here \( B u \) is regarded as being prescribed on \( \partial R_1 \).

That the functionals \( I(u) \), \( J(u,v) \) and \( K(v) \) provide a basis for complementary variational principles associated with (2.22) is made clear by the theorems we cite below. While our boundary conditions are slightly more general than those we have found elsewhere, these results are equivalent to those found in [13-15].

**Theorem 4.1.** The functional \( J(u,v) \) of (4.1) assumes a stationary value at the point \( (u^*, v^*) \) if and only if \( u^* \) is a solution of (2.23) and \( v^* \) is a solution of (2.24)

**Theorem 4.2.** Among all the elements \( u \) which satisfy the boundary conditions \( B(g - u) = 0 \) on \( \partial R_1 \) and for which there is some \( y \) such that \( y = Tu \), the unique element \( u^* \) such that \( T(Tu^*) + f = 0 \) in \( R \) and \( B^*(Tu^*) - S = 0 \) on \( \partial R_2 \), makes the functional \( I(u) \) of (4.2) assume its minimum value.

**Theorem 4.3.** Among all the elements \( v \) which satisfy the boundary conditions \( B^* v - S = 0 \) on \( \partial R_2 \) and for which \( T v + f = 0 \) in \( R \), the unique element \( v^* \)
such that \( T\bar{u}^* = \bar{v}^* \) and \( B(g - u^*) = 0 \) on \( \partial \Omega \), \( u^* \) being the solution of (2.22), makes the functional \( K(v) \) of (4.3) assume its maximum value.

**Partial Proofs and Comments:** While the proofs of these theorems are straightforward, some of the results obtained along the way are sufficiently valuable to warrant our furnishing a few details. Concerning 4.1, we find after a little algebra that if \( h \) and \( \bar{w} \) are arbitrary elements in \( U \) and \( \bar{V} \), and \( \bar{u} = u^* + \bar{h} \), \( \bar{v} = v^* + \bar{w} \), then

\[
J(u, \bar{v}) - J(u^*, \bar{v}^*) = (T\bar{v}^* + f, h) + (B\bar{v}^* - S, h)_{\partial \Omega} + \\
+ (T\bar{u}^* - \bar{v}^*, \bar{w}) + (B\bar{w}, g - u^*)_{\partial \Omega} - \\
- \frac{1}{2} [\bar{w}, \bar{w}] + [Th, \bar{w}] - (B\bar{w}, h)_{\partial \Omega}
\]

Clearly, if \( u^* \) and \( v^* \) satisfy (2.23) and (2.24), the first four terms on the right-hand side vanish and we conclude that \( (u^*, v^*) \) is a critical point of \( J(u, v) \). Likewise, if the pair \( (u^*, v^*) \) is a critical point of \( J(u, \bar{v}) \), then \( u^* \) and \( \bar{v}^* \) satisfy (2.23) and (2.24) weakly. Note that at a critical point,

(4.4)

\[
J(u, v) - J(u^*, \bar{v}^*) = - \frac{1}{2} \|v\|_{\bar{V}}^2 + [Th, \bar{w}] - (B\bar{w}, h)_{\partial \Omega}
\]

where \( \|\bar{v}\|_{\bar{V}}^2 = [\bar{w}, \bar{w}] \) and \( \bar{w} = v - v^* \), \( h = u - u^* \).

Following similar procedures in the case of \( I(u) \) and \( K(v) \) leads to

(4.5)

\[
I(u) = I(u^*) + \frac{1}{2} [Th, Th]
\]

656
where in (4.5) we have equated to zero terms \{T^*T_u + f, h\} + \{B^*T_u - S, h\} and in (4.6) we set equal to zero \([Tu - v, w] + \{w, B(g - u)\}\). In arriving at (4.6), we have used the identity, \{B^*v, u\} = \{w, Tu\}. In particular, admissible variations in \(u\) associated with \(I(u)\) are limited to the subspace \(\hat{U} \subseteq U\) such that \(\hat{u} \in \hat{U} \rightarrow B^*\hat{u} = 0\) on \(\partial \hat{R}_1\) and \(\hat{u} \in T(U)\); similarly, admissible variations in \(v\) associated with \(K(v)\) must be such that \(T^*v = 0\) in \(R\) and \(B^*v = 0\) on \(\partial \hat{R}_2\). From this latter observation we obtain the stated identity immediately.

It is clear from (4.5) and (4.6) that \(I(u) \geq I(u^*)\) and \(K(v) \leq K(v^*)\) since \(\|Th\|_V^2 > 0\) and \(\|v\|_V^2 > 0\). Consequently, \(I(u)\) assumes a global minimum at \(u^*\) and \(K(v)\) a global maximum at \(v^*\). Uniqueness of \(u^*\) and \(v^*\) can be shown to follow from the fact that \(-[v, f]/2 + \{f, u\}\) is concave in \(v\) and convex in \(u\); i.e., the generalized Hamiltonian has saddle-point behavior (see, for example, [10]).

The relationship between conjugate projections and variational formulations is established by the following theorem.

**Theorem 4.4.** Let \(\tilde{u} = Nu = \sum \alpha \phi_\alpha\) and \(\tilde{v} = Pv = \sum \Delta b_\Delta^\alpha\) denote projections of \(u\) and \(v\) into the subspaces \(M_G\) and \(M_H\) described earlier. Let \(I(\tilde{u}) = I(a^\alpha), J(\tilde{u}, \tilde{v}) = J(a^\alpha, b_\Delta),\) and \(K(\tilde{v}) = K(b_\Delta)\) denote the
restriction of the functionals \( I(u) \), \( J(u,y) \), and \( K(y) \) to the subspaces \( M_G \) and \( \mathcal{N}_H \). Then

1. Among all \( u \in M_G \), that which provides a minimum for \( I(u) \) is obtained by choosing the coefficients \( a^\alpha \) which satisfy (2.27).

2. Among all \( u \in M_G \) and \( v \in \mathcal{N}_H \), those which provide a stationary value to \( J(u,v) \) are obtained by choosing the coefficients \( a^\alpha \) and \( b_\Delta \) which satisfy (2.30) and (2.32).

3. Among all \( v \in \mathcal{N}_H \), that which provides a maximum for \( K(v) \) is obtained by choosing the coefficients \( b_\Delta \) which satisfy (2.37).

Proof: We simply note that

\[
\frac{\partial I(a^\mu)}{\partial a^\alpha} = \sum_k K_{a^\mu a^\alpha} + f_\alpha ; \quad \frac{\partial J(a^\mu, b_\Gamma)}{\partial a^\alpha} = \sum_\Delta T_\alpha^\Delta b_\Delta + f_\alpha ; \quad \frac{\partial J(a^\mu, b_\Gamma)}{\partial b_\Gamma} = \sum_\alpha a^\alpha T_\alpha^\Delta - \sum_\Gamma H_\Delta^\Gamma b_\Gamma ; \quad \frac{\partial K(b_\Gamma)}{\partial b_\Gamma} = -\sum_\Gamma H_\Delta^\Gamma b_\Gamma + g_\Delta .
\]

4.2 The hypercircle method. We shall conclude this section by establishing the relationship between the theory presented thusfar and the method of the hypercircle of Prager and Synge [11,12]. To make clear the essential differences in the two approaches, it is convenient to generalize the problem (2.1) slightly to

\[
T^* E u + f = 0 \quad \text{on } \partial R_1 ; \quad B^* E u - S = 0 \quad \text{on } \partial R_2
\]

where \( E \) is a strictly positive nonsingular operator, self-adjoint in the sense that \( [E w_1, w_2] = [w_1, E w_2] \) for every
\[ v_1, v_2 \in V \] (indeed, \( E \) may merely represent a positive variable coefficient in the governing equation and \( \{B^*v, u\}_{\partial R} = [1 \cdot u, v]_{\partial R} \). The corresponding canonical (Euler-Hamilton) forms and functionals are easily shown to be

\begin{align*}
(4.8) & \quad T_u = E^{-\frac{1}{2}}_v ; \ u - g = 0 \text{ on } \partial R_1 \\
(4.9) & \quad T^*v + f = 0 ; \ B^*v - S = 0 \text{ on } \partial R_2
\end{align*}

and

\begin{align*}
I(u) &= \frac{1}{2} \{ETu, Tu\} + \{f, u\} - \{S, u\}_{\partial R_2} \\
J(u, v) &= \{v, Tu\} - \frac{1}{2} \{v, E^{-\frac{1}{2}}_v\} + \{f, u\} + \\
&\quad + \{B^*v, g - u\}_{\partial R_1} - \{S, u\}_{\partial R_2} \\
K(v) &= -\frac{1}{2} \{v, E^{-\frac{1}{2}}_v\} + \{B^*v, u\}_{\partial R_1}
\end{align*}

Again, the last term in \( I(u) \) of (4.2) now vanishes \( B = 1 \), \( I(u) \) assumes a minimum value at the solution \( u^* \) of (4.7), \( K(v) \) is a maximum at the solution \( v^* \) of (4.9) and \( J(u, v) \) is stationary at \( (u^*, v^*) \).

The essence of the hypercircle approach is as follows: let \( W \) denote the set of all elements \( u \) that satisfy the boundary condition \( B(g - u) = 0 \text{ on } \partial R_1 \). Each \( u \in W \) is of the form \( u = u_0 + \hat{u} \), where \( Bu_0 = Bg \) and \( B\hat{u} = 0 \text{ on } \partial R_1 \). Let \( T(W) \) denote the image space of \( W \) under the mapping \( T \) and let \( X = ET(W) \times T(W) \) denote the set of ordered pairs

\begin{equation}
(4.11) \quad \Lambda = \{ETu, Tu\}
\end{equation}
Since each $\eta \in \mathcal{X}$ is of the form $\eta = \eta_0 + \hat{\eta}$, where $\eta_0 = (\mathcal{E}u_0, \mathcal{U}u_0)$ and $\hat{\eta} = (\mathcal{E}u, \mathcal{U}u)$, $\mathcal{X} = \eta_0 + \hat{\mathcal{X}}$, where $\hat{\mathcal{X}}$ is a linear space. Similarly, let $\mathcal{W}$ denote the set of all $\mathcal{Y}$ that satisfy $T^*\mathcal{Y} = -f$ in $\mathcal{R}$ and $B^*\mathcal{Y} = S$ on $\partial \mathcal{R}_1$. Each $\mathcal{Y} \in \mathcal{W}$ is of the form $\mathcal{Y} = \mathcal{Y}_0 + \mathcal{Y}',$ where $T^*\mathcal{Y}_0 = -f$, $T^*\mathcal{Y} = 0$ in $\mathcal{R}$; $B^*\mathcal{Y}_0 = S$, $B^*\mathcal{Y} = 0$ on $\partial \mathcal{R}_2$. The set $\mathcal{Y} = \mathcal{W} \times E^{-1}\mathcal{W}$ consists of ordered pairs

\[(4.12) \quad \mathcal{Y} = (\mathcal{Y}, E^{-1}\mathcal{Y})\]

which can be decomposed into $\mathcal{Y}_0 + \mathcal{Y}'$, where $\mathcal{Y}_0 = (\mathcal{Y}_0, E^{-1}\mathcal{Y}_0)$, $\mathcal{Y}' = (\mathcal{Y}', E^{-1}\mathcal{Y}')$. Again $\mathcal{Y} = \mathcal{Y}_0 + \mathcal{Y}'$, where $\mathcal{Y}'$ is a linear space. If $u^*, v^*$ are the solutions of (4.8) and (4.9) there is a unique point $\mathcal{X}$ of intersection of $\mathcal{X}$ and $\mathcal{Y}$:

\[(4.13) \quad \mathcal{X} \cap \mathcal{Y} = \mathcal{X} = (v^*, Tu^*)\]

Further, if we suppose that $\mathcal{X}, \mathcal{Y} \subset \mathcal{Z}$; then an inner product $\langle\cdot, \cdot\rangle$ can be introduced on $\mathcal{Z}$ according to the rule that if $\mathcal{A}_A$ is the ordered pair $(\mathcal{A}_A, \mathcal{B}_A)$, $\mathcal{A}, \mathcal{B} \in \mathcal{Y}$, then

\[(4.14) \quad \langle A_A, B_B \rangle = [A_A, B_B]\]

Here $[\cdot, \cdot]$ is the inner product on $\mathcal{Y}$ used previously. Symmetry of $\langle A_A, B_B \rangle$ is provided through the assumption that $E$ is self adjoint. The associated norm is $\|A\|^2 = \langle A_A, A_A \rangle$ and the squared distances from the solution pair $\mathcal{X}$ and arbitrary elements $\mathcal{A} \in \mathcal{X}$ and $\mathcal{Y} \in \mathcal{Y}$ are

\[(4.15) \quad d_{x\mathcal{X}}^2 = \|x - \mathcal{X}\|^2; \quad d_{x\mathcal{Y}}^2 = \|x - \mathcal{Y}\|^2\]

660
The hypercircle method consists of seeking approximations to \( X \) in \( X \) and \( Y \) by minimizing \( d_{xa}^2 \) and \( d_{xi}^2 \) among finite dimensional subspaces of \( X \) and \( Y \). Approximations are sought in finite dimensional subspaces of \( X \) and \( Y \), and the approximation is improved by computing the center of a hypercircle in \( Z \), the diameter of which is defined by approximate solutions in \( X \) and \( Y \).

The relationship between the above concepts and the theory presented earlier is made clear by the following theorems.

**Theorem 4.5.** Minimization of \( d_{xa}^2 \) of (4.16) is equivalent to minimizing \( I(u) \) of (4.10). In fact,

\[
(4.16) \quad d_{xa}^2 = 2I(u) + \text{constant}
\]

**Theorem 4.6.** Minimization \( d_{xi}^2 \) of (4.15) is equivalent to minimizing \( -K(y) \) of (4.10). In fact

\[
(4.17) \quad d_{xi}^2 = -2K(y) + \text{constant}
\]

**Proofs:** Clearly,

\[
d_{xa}^2 = \| x - \lambda \|^2 = [v^* - ETu, Tu^* - Tu]
\]

\[
= [v^*, Tu^*] + [ETu, Tu] - [ETu, Tu^*] - [v, Tu]
\]

But, owing to the self-adjoint property of \( E \), \([ETu, Tu^*] = [Tu, ETu^*] = [Tu, v^*] \); moreover, \([v^*, Tu^*]\), for fixed \((u^*, v^*)\), is a constant. Thus, Theorem 4.5 follows from the relation...
\[ d^2_{x\Gamma} = [E Tu, Tu] + 2\{f, u\} - 2\{S, u\} \_\_ R_2 + \text{const.} \]
\[ = 2I(u) + \text{const.} \]

Similarly, since \( E^{-1} \) is self-adjoint,

\[ d^2_{x\Gamma} = \|X - T\|^2 = [v^*, Tu^*] + [v, E^{-1} v] - [v, Tu^*] \]
\[ - [v^*, E^{-1} v] \]
\[ = [v, E^{-1} v] - 2\{B^*, g\} \_\_ R_2 + \text{const.} \]
\[ = -2K(v) + \text{const.} \]

which proves Theorem 4.6.


While a detailed study of accuracy and convergence of mixed finite element models is, because of space limitations, postponed for a future paper, we shall outline here preliminary results obtained for certain special cases.

5.1 Primal and dual problems. Error estimates for approximations to the primal and dual problems follow the usual procedure. We denote by \( h \) the maximum diameter of a finite element in the mesh \( \mathcal{K} \) and by \( u^*, v^* \) the solution pair of (2.23) [or (2.22)] and (2.24). If \( \tilde{U} \in M_G \) and \( \tilde{v} \in N_H \) are interpolants of \( u^*, v^* \), then we can obtain the interpolation formulas [6]

\[ |D_\alpha (u^* - \tilde{U})| \leq Kh^r \quad ; \quad |D_\alpha (v^* - \tilde{v})| \leq Ch^s \]

as \( h \to 0 \) where \( K \) and \( C \) are constants independent of \( h \) and \( r, s > 0 \). For example, if \( \tilde{U} \) and \( \tilde{v} \) are complete
polynomials in $x$ of order $p$ and $q$ respectively, then
$$r = p + 1 - |a| \quad \text{and} \quad s = q + 1 - |a|.$$ If $W \in M_G$ minimizes $I(u)$ among all $U \in M_G$ and if $v \in N_H$ maximizes $K(y)$ among all $v \in N_H$ (i.e., if $W$ and $v$ are the finite element approximations of the primal and dual problems) then
\[
|I(W) - I(u^*)| = \frac{1}{2}[T(W - u^*),T(W - u^*)] \leq |I(U) - I(u^*)|
\]
\[
(5.2) \quad = \frac{1}{2}[T(U - u^*),T(U - u^*)]
\]
and
\[
|K(v) - K(v^*)| = \frac{1}{2}\|v_l - v^*_l\|_y^2
\]
\[
(5.3) \quad \leq |K(v) - K(v^*)|
\]
\[
= \frac{1}{2}\|v_l - v^*_l\|_y^2
\]
Thus, if, for example, $T$ is a partial differential operator of order $m$, we have as $h \to 0$
\[
(5.4) \quad \|W - u^*\| \leq Kh^{p+1-m} \quad \|v_l - v^*_l\| \leq Ch^{q+1}
\]
where we have used the notation $\{u,\tilde{u}\} = \|u\|^2$ and $\{v,\tilde{v}\} = \|v\|^2 = \|v\|_y^2$; in (5.4), $\|\tilde{u}\| = \|u\|\infty_R$.

5.2 Mixed problems. The question of convergence and accuracy of mixed finite element approximations is more difficult, and we shall be content here to establish sufficient conditions for convergence for certain special cases. In particular, we consider the class of problems in which $u$ and the finite element method approximations
satisfy homogeneous boundary conditions on \( \partial R_1 \). Thus, if \((U^*, \nu^*)\) denote the mixed finite-element approximate solutions of the canonical problem (2.23) and (2.24), then

\[
\begin{align*}
\{T^*\nu^* + f, U\} &= 0 ; \quad \{TU^* - \nu^*, \nu\} = 0 \\
\{B^*\nu^* - S, U\}\mid_{\partial R_2} &= 0 ; \quad \{B^*\nu, U\}\mid_{\partial R_1} = 0
\end{align*}
\]

for all \( U \in M_G \subset \mathcal{U} \) and \( \nu \in N_H \subset \mathcal{V} \), with all \( U = 0 \) on \( \partial R_1 \).

In most instances, \( M_G \) will consist of complete piecewise polynomials of degree \( q \), and \( T \) is a partial differential operator of order \( m \); then we shall also assume that there exist interpolants in \( M_G \) and \( N_H \) such that at each \( x \in \mathcal{R} \),

\[
|u^* - \tilde{u}| \leq K_1h^{p+1}
\]

(5.6) \[
|T(u^* - \tilde{U})| \leq K_2h^{p+1-m}
\]

|\nu^* - \tilde{\nu}| \leq K_3h^{q+1}

If \( e_u \) and \( e_v \) denote the errors in \( U^* \) and \( \nu^* \); i.e.,

(5.7) \[
e_u = u^* - U^* ; \quad e_v = \nu^* - \nu^*
\]

then we wish to show that each error vanishes in some normed sense as \( h \to 0 \).

Our first result involves \( \|e_v\| = \|e_v\|_{\mathcal{V}} = [e_v, e_v] \) and rests on the following relations:
Lemma 5.1. Let $J(u, v)$ denote the bilinear functional in (4.1) and let (5.5) hold. Then the following inequalities hold:

\begin{align}
(5.8) \quad (i) \quad J(U, V^*) - J(U^*, V^*) & = [e_v, T(u^* - U)] - \frac{1}{2} \| e_v \|^2 \\
(5.9) \quad (ii) \quad J(U, V^*) & \geq J(U^*, V) \\
(5.10) \quad (iii) \quad J(U^*, V^*) + [e_v, T(u^* - U)] & \geq J(U, V^*) \\
& \geq J(U^*, V) 
\end{align}

Partial proof: These relations follow directly from the easily verified identity,

\begin{align}
J(u_1, v_1) - J(u_2, v_2) & = \{T^* v_2 + f, u_1 - u_2\} + \\
& + [v_1 - v_2, Tu_1 - v_2] - \frac{1}{2} \| v_1 - v_2 \|^2 + \\
& + \{B^* v_2 - S, u_1 - u_2\}_R + \\
& + \{B^* (v_1 - v_2), g - u_1\}_1 + \\
& + \{B^* (v_1 - v_2), g - u_2\}_2 + \\
& + [v_1 - v_2, Tu_1 - v_2] - \frac{1}{2} \| v_1 - v_2 \|^2 + \\
& + [v_1 - v_2, Tu_1 - v_2] - \frac{1}{2} \| v_1 - v_2 \|^2 + \\
& [v_1 - v_2, Tu_1 - v_2] + [v_1 - v_2, T(u^* - U)] = 0 
\end{align}

To obtain (5.9), we use (5.11) and the fact that, generally,

\begin{align}
[v - v^*, Tu - u^*] + [v - v^*, T(u^* - U)] = 0 
\end{align}

This result and (5.9) hold for the general (nonhomogeneous) boundary conditions on $\partial R_1$ and $\partial R_2$.

Collecting results (5.9) - (5.10) and using the fact that $J(U^*, V) - J(U^*, V^*) = [v - v^*, T(u^* - u^*)] - \frac{1}{2} \| v - v^* \|^2$, leads directly to the convergence theorem.
Theorem 5.1. Let (5.5) hold. Then

\[
\|e_v\|^2 \leq \|v^* - \bar{v}\|^2 + 2\|v^* - \bar{v}\| \| T(e_u) \| + \\
+ 2\| T(u^* - U) \| \| e_v \|
\]  

(5.12)

This result establishes the fact that \( \|e_v\| \to 0 \) as \( h \to 0 \) if interpolation estimates of the type (5.6) hold. For example, if we set \( \bar{v} = \bar{v} \) in (5.12) and make use of (5.6), then

\[
\|e_v\| \leq K_1h^{2(q+1)} + 2K_3h^{2(q+1)} \| T(e_u) \| + \\
+ 2K_2\|e_v\|n^{p+1-m}
\]  

(5.13)

It appears that nothing much sharper than (5.12) or (5.13) can be said until some estimate on \( \|Te_u\| \) is established. While we shall reserve a more detailed discussion of this question for a later paper, we shall show that \( Te_u \to 0 \) at least weakly for a wide range of problems.

Returning to (5.5), let \( W^* \) denote the usual "displacement type" finite-element approximation of the primal problem subject to the same boundary conditions on \( \partial R \) as \( U^* \), and for convenience, suppose that \( U^* \) and \( W^* \) belong to the same subspace \( M_G \). Then we can derive the remarkably similar pair of orthogonality conditions,

\[
[TW^* - V^*, U] = 0 \quad ; \quad [TU^* - \bar{V}^*, V] = 0
\]  

(5.14)

The first member of the pair arises due to the fact that \( [T^*T(W^*), U] = -\{f, U\} = [T^*V, U] \) and our assumption concern-
GENERALIZED CONJUGATE FUNCTIONS

ing the boundary conditions. As a measure of the distance ("angle") between the subspaces \( T(\xi G) \subset \xi \) and \( N_H \subset \xi \), we introduce

\[
(5.15) \quad \xi = TU - V
\]

If we can find an interpolant \( \tilde{T}U \) in \( T(\xi G) \) such that

\[
|\tilde{T}U - Tu^*| \leq K_4 h^n \quad \text{(generally } n = p+1-m)\]

then, by use of (5.6) and the triangle inequality, there is an element \( \tilde{\omega} \in \xi \) such that as \( h \rightarrow 0 \)

\[
(5.16) \quad \|\tilde{\omega}\| \leq K h^r ; \quad r = \min\{n, q+1\}
\]

Now substituting (5.15) into (5.14) and use of (5.14) leads to

\[
[TW^* - TU^*, V] = [T(W^* - u^*) - Te_u, V] = [TW^* - TU^*, \tilde{\omega}]
\]

Since the \( \xi \) are generated by bases whose linear combinations are dense in \( \xi \), and since use of the Schwarz inequality leads to

\[
(5.17) \quad [T(W^* - u^*) - Te_u, \tilde{V}] \leq \|TW^* - TU^*\| \cdot Kh^r
\]

then \( TU^* \) at least converges weakly to \( Tu^* \) as \( h \rightarrow 0 \), assuming that \( \|TW^* - V^*\| \) is bounded.

5.3 A posteriori estimates. We conclude our investigation with the observation that some of the results of Aubin and Burchard [17] can be adjusted to obtain a posteriori estimates for certain special cases of our approximations.

In particular, if we replace \( f(x) \) in all our results by \( cu(x) + f(x) \), \( c \) being a positive constant, and if we introduce the rather severe assumption that \( U \) and \( V \)
satisfy the boundary conditions on $\partial R$, and, further, if $TT$ is m-elliptic in the sense that

$$a(u,u) \equiv [Tu,Tu] \geq \gamma^2 \|u\|_m^2$$

then it is easy to show that

$$K\|u^*-U\|_m^2 + c\|u^*-U\|^2 \leq K^{-1}\|V-Tu\|_V^2 + c^{-1}\|f-T\gamma^*-cU\|$$

By setting $U = U^*$, $V = V^*$, we immediately obtain a bound on $\|e_u\|_m^2 + K'\|e_u\|^2$. If $c = 0$, the same procedure used to get (5.19) leads to

$$a(u^*-U,u^*-U) = [T^*V^*-T^*U,u^*-U] + [V-TU,T(u^*-U)]$$

which leads to the less useful result,

$$K\|u^*-U\|^2 \leq \|V-Tu\|_V^2 + \|V^*-V\|_V \|V-TU\|_V + \|T(u^*-U)\|_V$$

Acknowledgment: In recent years, my work on finite elements has been supported by the Air Force Office of Scientific Research under Contract F44620-69-C-0124. This support is gratefully acknowledged. It is also a pleasure to thank Mr. J. N. Reddy for assistance in proofreading the paper, checking certain derivations, and many helpful discussions.

REFERENCES


GENERALIZED CONJUGATE FUNCTIONS

(1971), 317-325.


