On compact operators

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Abstract

In this basic note, we consider some basic properties of compact operators. We also consider the spectrum of compact operators on Hilbert spaces. A basic numerical example involving a compact integral operator is provided for further illustration.

1 Introduction

The goal of this brief note is to collect some of the basic properties of compact operators on normed linear spaces. The results discussed here are all standard and can be found in standard references such as [3, 7, 6, 5, 1, 4] to name a few. The point of this note is to provide an accessible introduction to some basic properties of compact operators. After stating some preliminaries and basic definitions in Section 2, we follow up by discussing some examples of compact operators in Section 3. Then, in Section 4, we discuss the range space of a compact operator, where we will see that the range of a compact operator is “almost finite-dimensional”. In Section 5, we discuss a basic result on approximation of compact operators by finite-dimensional operators. Section 6 recalls some basic facts regarding the spectrum of linear operators on Banach spaces. Sections 7–8 are concerned with spectral properties of compact operators on Hilbert spaces, Fredholm’s theorem of alternative, and Spectrum Theorem for compact selfadjoint operators. Finally a numerical example is presented in Section 9.

2 Preliminaries

Let us begin by recalling the notion of precompact and relatively compact sets.

Definition 2.1. (Relatively Compact)
Let $X$ be a metric space; $A \subseteq X$ is relatively compact in $X$, if $\bar{A}$ is compact in $X$.

Definition 2.2. (Precompact)
Let $X$ be a metric space; $A \subseteq X$ is precompact (also called totally bounded) if for every $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_N$ in $A$ such that $\bigcup_{i=1}^{N} B(x_i, \epsilon)$ covers $A$.

The following Theorem shows that when we are working in a complete metric space, precompactness and relative compactness are equivalent.

Theorem 2.3. Let $X$ be a metric space. If $A \subseteq X$ is relatively compact then it is precompact. Moreover, if $X$ is complete then the converse holds also.

Then, we define a compact operator as below.

Definition 2.4. Let $X$ and $Y$ be two normed linear spaces and $T : X \to Y$ a linear map between $X$ and $Y$. $T$ is called a compact operator if for all bounded sets $E \subseteq X$, $T(E)$ is relatively compact in $Y$.

By Definition 2.4, if $E \subset X$ is a bounded set, then $T(E)$ is compact in $Y$. The following basic result shows a different ways of looking at compact operators.

Theorem 2.5. Let $X$ and $Y$ be two normed linear spaces; suppose $T : X \to Y$, is a linear operator. Then the following are equivalent.

1. $T$ is compact.
2. The image of the open unit ball under $T$ is relatively compact in $Y$.
3. For any bounded sequence $\{x_n\}$ in $X$, there exist a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ that converges in $Y$.

Let us denote by $B[X]$ the set of all bounded linear operators on a normed linear space $X$:

$$B[X] = \{ T : X \to X | T \text{ is a bounded linear transformation} \}.$$ 

Note that equipped by the operator norm $B[X]$ is a normed linear space. It is simple to show that compact operators form a subspace of $B[X]$. The following result (cf. [6] for a proof) shows that the set of compact operators is in fact a closed subspace of $B[X]$.

**Theorem 2.6.** Let $\{T_n\}$ be a sequence of compact operators on a normed linear space $X$. Suppose $T_n \to T$ in $B[X]$. Then, $T$ is also a compact operator.

Another interesting fact regarding compact linear operators is that they form an ideal of the ring of bounded linear mappings $B[X]$; this follows from the following result.

**Lemma 2.7.** Let $X$ be a normed linear space, and let $T$ and $S$ be in $B[X]$. If $T$ is compact, then so are $ST$ and $TS$.

**Proof.** Consider the mapping $ST$. Let $\{x_n\}$ be a bounded sequence in $X$. Then, by Theorem 2.5(3), there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ that converges in $X$:

$$\lim_{k \to \infty} Tx_{n_k} = y^* \in X.$$

Now, since $S$ is continuous, it follows that $STx_{n_k} \to S(y^*)$; that is, $\{STx_{n_k}\}$ converges in $X$ also, and so $ST$ is compact. To show $TS$ is compact, take a bounded sequence $\{x_n\}$ in $X$ and note that $\{Sx_n\}$ is bounded also (since $S$ is continuous). Thus, again by Theorem 2.5(3), there exists a subsequence $\{TSx_{n_k}\}$ which converges in $X$, and thus, $TS$ is also compact.

**Remark 2.8.** A compact linear operator of an infinite dimensional normed linear space is not invertible in $B[X]$. To see this, suppose that $T$ has an inverse $S$ in $B[X]$. Now, applying the previous Lemma, we get that $I = TS = ST$ is also compact. However, this implies that the closed unit ball in $X$ is compact, which is not possible since we assumed $X$ is infinite dimensional.$^1$

We end this section by a useful result regarding compact operators on Hilbert spaces. Let us recall that a sequence $\{x_n\} \subset H$ is said to converge weakly to $x \in H$ if, for every $v \in H$,

$$\langle x_n - x, v \rangle \to 0 \quad \text{as} \quad n \to \infty.$$ 

We denote weak convergence of $\{x_n\}$ to $x$ by $x_n \rightharpoonup x$. The following result states that a bounded linear operator on a Hilbert space $H$ is compact if and only if it maps weakly convergent sequences to strongly convergent ones.

**Theorem 2.9.** Let $H$ be a Hilbert space and $T \in B[H]$. Then, the following are equivalent.

1. $T$ is compact.
2. Let $\{x_n\} \subset H$ be a sequence such that $x_n \to x$ in $H$. Then, $Tx_n \to Tx$, as $n \to \infty$, in norm.

**Remark 2.10.** The above result has the following immediate consequence. Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal set in $H$. Then,

$$Te_n \to 0, \quad \text{as} \quad n \to \infty. \quad \text{(2.1)}$$

This is seen by noting that for every $x \in H$, the Parseval identity says that $\sum_{j=1}^\infty |\langle x, e_j \rangle|^2 = \|x\|^2 < \infty$. Therefore, it must be the case that $\langle x, e_j \rangle \to 0$, as $n \to \infty$. Since this holds for every $x \in H$, we have that $e_j \rightharpoonup 0$. Therefore, (2.1) follows from Theorem 2.9.

### 3 Some examples of compact operators

Here we consider two special instances of compact operators: the finite-dimensional (or finite-rank) operators, and the Hilbert-Schmidt operators.
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Finite-dimensional operators Let \( T : X \to Y \) be a continuous linear mapping between normed linear spaces If the range space \( \text{Ran}(T) \) is of finite dimension, \( \dim(\text{Ran}(T)) < \infty \), we call \( T \) a finite-dimensional operator. It is straightforward to see that finite-dimensional operators are compact. This is seen by noting that for a bounded set \( E \subseteq X \), \( \overline{T(E)} \) is closed and bounded in the finite-dimensional subspace \( \text{Ran}(T) \subseteq Y \). Therefore, Heine-Borel Theorem applies, and \( \overline{T(E)} \) is compact in \( \text{Ran}(T) \subseteq Y \).

Hilbert-Schmidt operators Let \( D \subset \mathbb{R}^n \) be a bounded domain. We call a function \( k : D \times D \to \mathbb{R} \) a Hilbert-Schmidt kernel if

\[
\int_D \int_D |k(x,y)|^2 \, dx \, dy < \infty,
\]

that is, \( k \in L^2(D \times D) \) (note that one special case is when \( k \) is a continuous function on \( D \times D \)). Define the integral operator \( K \) on \( L^2(D) \), \( K : u \to Ku \) for \( u \in L^2(D) \), by

\[
[Ku](x) = \int_D k(x,y)u(y) \, dy.
\]

Clearly, \( K \) is linear; moreover, it is simple to show that \( K : L^2(D) \to L^2(D) \):

Let \( u \in L^2(D) \), then

\[
\int_D \left( |(Ku)(x)| \right)^2 \, dx = \int_D \int_D |k(x,y)u(y)|^2 \, dy \, dx \\
\leq \int_D \left( \int_D |k(x,y)|^2 \, dy \right) \left( \int_D |u(y)|^2 \, dy \right) \, dx \quad \text{(Cauchy-Schwarz)}
\]

so that \( Ku \in L^2(D) \). The mapping \( K \) is what we call a Hilbert-Schmidt operator.

**Lemma 3.1.** Let \( D \) be a bounded domain in \( \mathbb{R}^n \) and let \( k \in L^2(D \times D) \) be a Hilbert-Schmidt kernel. Then, the integral operator \( K : L^2(D) \to L^2(D) \) given by \( [Ku](x) = \int_D k(x,y)u(y) \, dy \) is a compact operator.

The basic idea of the proof is to write the operator \( K \) as a limit of finite-dimensional operators and then apply Theorem 2.6.

**Remark 3.2.** One can think of Hilbert-Schmidt operators as generalizations of the idea of matrices to infinite-dimensional spaces. Note that if \( A \) is an \( n \times n \) matrix (a linear mapping on \( \mathbb{R}^n \)), then, the action \( Ax \) of \( A \) on a vector \( x \in \mathbb{R}^n \) is given by

\[
[Ax]_i = \sum_{j=1}^n A_{ij}x_j \quad (3.2)
\]

Now note that,

\[
[Ku](x) = \int_D k(x,y)u(y) \, dy
\]

is an analog of (3.2) with the summation replaced with an integral.

4 Range of a compact operator

We saw in the previous section that a continuous linear map with a finite-dimensional range is compact. While the converse is not true in general, we can say something to the effect that the range of a compact operator is “almost finite-dimensional”. More precisely, the range of compact operators can be approximated by a finite-dimensional subspace within a prescribed \( \varepsilon \)-distance, as described in the following result. The proof given below follows that of [6] closely.

**Theorem 4.1.** Let \( T : X \to Y \) be a compact linear transformation between Banach spaces \( X \) and \( Y \). Then, given any \( \varepsilon > 0 \), there exists a finite-dimensional subspace \( M \) in \( \text{Ran}(T) \) such that, for any \( x \in X \),

\[
\inf_{m \in M} \|Tx - m\| \leq \varepsilon \|x\|.
\]
Proof. Let \( \varepsilon > 0 \) be fixed but arbitrary. Let \( B_X \) denote the closed unit ball of \( X \). Note that \( T(B_X) \) is precompact, and thus can be covered with a finite cover, \( \cup_{i=1}^N B(y_i, \varepsilon) \) with \( y_i \in T(B_X) \subseteq \text{Ran}(T) \). Now let \( M \) be the span of \( y_1, \ldots, y_N \), and note that \( M \subseteq \text{Ran}(T) \) and \( \text{dist}(Tz, M) \leq \varepsilon \) for any \( z \in B_X \). Therefore, for any \( x \in X \),

\[
\inf_{m \in M} \| T\left( \frac{x}{\|x\|} \right) - m \| \leq \varepsilon.
\]

And thus,

\[
\inf_{m \in M} \| T(x) - m' \| \leq \varepsilon \|x\|, \quad m' = m \|x\|, \quad m \in M.
\]

\( \square \)

5 Approximation by finite-dimensional operators

We have already noted that finite-dimensional operators on normed linear spaces are compact. Moreover, we know by Theorem 2.6 that the limit (in the operator norm) of a sequence of finite-dimensional operators is a compact operator. Moreover, we have seen that the range of compact operators can be approximated by a finite-dimensional subspace, in the sense described in Theorem 4.1. A natural follow up is the following question: Let \( T : X \to Y \) be a compact operator between normed linear spaces \( X \) and \( Y \), is it then true that \( T \) is a limit (in operator norm) of a sequence of finite-dimensional operators? The answer to this question, is negative for Banach spaces in general (see [2]). However, the result holds in the case \( Y \) is a Hilbert space, as given by the following known result:

**Theorem 5.1.** Let \( T : X \to Y \) be a compact operator, where \( X \) is a Banach space, and \( Y \) is a Hilbert space. Then, \( T \) is the limit (in operator norm) of a sequence of finite-dimensional operators.

**Proof.** Let \( B_X \) denote the closed unit ball of \( X \). Since \( T \) is compact, we know \( T(B_X) \) is precompact; thus for any \( n \geq 1 \) there exists \( y_1, \ldots, y_N \) in \( T(B_X) \) such that \( T(B_X) \subseteq \cup_{i=1}^N B(y_i, 1/n) \). Let \( M_n = \text{span}\{y_1, \ldots, y_N\} \) and let \( \Pi_n \) be the orthogonal projection of \( Y \) onto \( M_n \). First note that for any \( y \in Y \), we have

\[
\|\Pi_n(y) - y_i\| \leq \|y - y_i\|, \quad i = 1, \ldots, N.
\]

Next define the mapping \( T_n \) by

\[
T_n = \Pi_n \circ T.
\]

We know, by construction, for any \( x \in B_X \), \( \|Tx - y_i\| \leq 1/n \) for some \( i \in \{1, \ldots, N\} \). Moreover, by (5.1),

\[
\|T_n x - y_i\| = \|\Pi_n(Tx) - y_i\| \leq \|Tx - y_i\| \leq 1/n.
\]

Therefore, for any \( x \in X \), with \( \|x\| \leq 1 \),

\[
\|T_n x - y_i\| = \|Tx - T_n x\| \leq \|Tx - y_i\| + \|y_i - T_n x\| \leq 1/n + 1/n = 2/n,
\]

and thus, \( \|T - T_n\| \leq 2/n \to 0 \) as \( n \to \infty \). \( \square \)

6 Spectrum of linear operators on a Banach space

Recall that for a linear operator \( A \) on a finite dimensional linear space, we define its spectrum \( \sigma(A) \) as the set of its eigenvalues. On the other hand, for a linear operator \( T \) on an infinite dimensional (complex) normed linear space the spectrum \( \sigma(T) \) of \( T \) is defined by,

\[
\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B[X] \},
\]

and \( \sigma(T) \) is the disjoint union of the point spectrum \( \sigma_p(T) \), (set of eigenvalues), continuos spectrum, \( \sigma_c(T) \), and residual spectrum, \( \sigma_r(T) \). Let us recall that the continous spectrum is given by,

\[
\sigma_c(T) = \{ \lambda \in \mathbb{C} : \text{Ker}(T - \lambda I) = \{0\}, \text{Ran}(T - \lambda I) \neq X, \text{Ran}(\overline{T - \lambda I}) = X \},
\]

and residual spectrum of \( T \) is given by,

\[
\sigma_r(T) = \{ \lambda \in \mathbb{C} : \text{Ker}(T - \lambda I) = \{0\}, \text{Ran}((T - \lambda I) \neq X \}.
\]
7 Some spectral properties of compact operators on Hilbert spaces

As we saw in Remark 2.8, a compact operator $T$ on an infinite dimensional normed linear space $X$ cannot be invertible in $B[X]$; therefore, we always have $0 \in \sigma(T)$. However, in general, not much can be said on whether $0$ is in point spectrum (i.e. an eigenvalue) or the other parts of the spectrum. However, we mention, the following special case:

**Lemma 7.1.** Let $H$ be a complex Hilbert space, and let $T \in B[H]$ be a one-to-one compact selfadjoint operator. Then, $0 \in \sigma_r(T)$.

**Proof.** Since zero is not in the point spectrum, it must be in $\sigma_c(T)$ or in $\sigma_r(T)$. We show $0 \in \sigma_r(T)$ by showing that the range of $T$ is dense in $H$. We first claim that $\text{Ran}(T) = \{0\}$. To show this we proceed as follows. Let $z \in \text{Ran}(T) = \{0\}$, and note that for every $x \in H$,

$$0 = \langle Tx, z \rangle = \langle x, Tz \rangle.$$ 

Hence, $Tz = 0$ which, since $T$ is one-to-one, implies that $z = 0$. This shows that $\text{Ran}(T) = \{0\}$. Thus, we have,

$$\text{Ran}(T) = (\text{Ran}(T)) = \{0\} = H.$$ 

The next result sheds further light on the spectrum of a compact operator on a complex Hilbert space.

**Lemma 7.2.** Let $T$ be a compact operator on a complex Hilbert space $H$. Suppose $\lambda$ is a non-zero complex number. Then,

1. $\text{Ker}(T - \lambda I)$ is finite dimensional.
2. $\text{Ran}(T - \lambda I)$ is closed.
3. $T - \lambda I$ is invertible if and only if $\text{Ran}(T - \lambda I) = H$.

**Proof.** The proof of this result is standard. See e.g. [6] for a proof. Here we just give the proof for the first statement of the theorem. Let $M = \text{Ker}(T - \lambda I)$. Note that since $T$ is continuous $M$ is closed. Also, note that $T |_{M} = \lambda I$. We show $M$ is finite-dimensional by showing that any bounded sequence in $M$ has a convergent subsequence. Take a bounded sequence $\{x_n\}$ in $M$. Then, there exists a subsequence $\{T x_{n_k}\}$ that converges. However, $T x_{n_k} = \lambda x_{n_k}$, thus (also using $\lambda \neq 0$), it follows that $\{x_n\}$ has a convergent subsequence.

Note that by the above theorem it follows immediately that a compact operator on a complex Hilbert space has empty non-zero continuous spectrum. This statement can be refined further by showing that in fact the same holds for the residual spectrum $\sigma_r(T)$ of a compact operator $T$ on a complex Hilbert space. That is, if a non-zero $\lambda \in \mathbb{C}$ is in $\sigma(T)$ then it must be an eigenvalue of $T$. This follows from the following well-known theorem, known has Fredholm’s theorem of alternative, which also has great utility in applications to integral equations:

**Theorem 7.3.** Let $H$ be a complex Hilbert space, and let $T \in B[H]$ be compact and let $\lambda$ be a nonzero complex number. Then, exactly one of the followings hold:

1. $T - \lambda I$ is invertible
2. $\lambda$ is an eigenvalue of $T$.

**Proof.** Assume that the first statement in not true. We show that $\lambda \in \sigma_p(T)$. We have already seen that $\lambda$ cannot be in continuous spectrum of $T$. Suppose now that $\lambda \in \sigma_r(T)$. Then, $\text{Ran}(T - \lambda I) \neq H$. Thus, there is a non-zero $x \in \text{Ran}(T - \lambda I)$, $\text{Ker}(T^* - \lambda I)$. That is, $\lambda \in \sigma_p(T^*)$. Now, by Lemma 7.2, $\text{Ran}(T^* - \lambda I)$ is closed and since $T^* - \lambda I$ is not invertible, by the same Lemma, $\text{Ran}(T^* - \lambda I) \neq H$. Therefore,

$$\text{Ker}(T - \lambda I) = \text{Ran}(T^* - \lambda I)^{\perp} \neq \{0\}.$$ 

That is, $\lambda \in \sigma_p(T)$, contradicting the supposition that $\lambda \in \sigma_r(T)$. Therefore, it follows that $\lambda \in \sigma_p(T)$. 

\[\square\]
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8 Spectral theorem for compact selfadjoint operators

Compact selfadjoint operators on infinite dimensional Hilbert spaces resemble many properties of the symmetric matrices. Of particular interest is the spectral decomposition of a compact selfadjoint operator as given by the following (see e.g., [6] for more details):

**Theorem 8.1.** Let $H$ be a (real or complex) Hilbert space and let $T : H \to H$ be a compact selfadjoint operator. Then, $H$ has an orthonormal basis $\{e_i\}$ of eigenvectors of $T$ corresponding to eigenvalues $\lambda_i$. In addition, the following holds:

1. The eigenvalues $\lambda_i$ are real having zero as the only possible point of accumulation.
2. The eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.
3. The eigenspaces corresponding to non-zero eigenvalues are finite-dimensional.

The eigenvalues of a compact selfadjoint operator can be ordered according to

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots.$$

Using the spectral theorem, we can “diagonalize” a compact selfadjoint operator $T : H \to H$, as follows,

$$Tu = \sum_{j=1}^{\infty} \lambda_j \langle u, e_j \rangle e_j, \quad u \in H.$$  

If $T$ has finitely many nonzero eigenvalues, then we say $T$ is finite-rank. In the case the set of nonzero eigenvalues of $T$ is countably infinite, we have that $\lambda_j \to 0$, as $j \to \infty$. To see the latter, we proceed as follows. We know by Remark 2.10 that

$$Te_j \to 0, \text{ as } j \to \infty. \quad (8.1)$$

On the other hand, if we assume that $\lambda_j$ do not converge to zero, then there is $\varepsilon > 0$ such that $|\lambda_j| \geq \varepsilon$, for all $j$. Thus, for every $j$

$$\|Te_j\| = |\lambda_j e_j| = |\lambda_j| \geq \varepsilon > 0,$$

which is in contradiction with (8.1). Therefore, it must be that $\lambda_j \to 0$, as $j \to \infty$.

In the case of a positive compact selfadjoint operator, we know that the eigenvalues are non-negative. Hence, we may order the eigenvalues as follows

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0.$$

For a strictly positive compact selfadjoint operator we can write the inverse operator by

$$T^{-1}u = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle u, e_j \rangle e_j.$$

This is a densely defined unbounded operator. It is densely defined because as we saw before range of $T$ is dense in $H$. The unboundedness follows from the earlier result that states compact operators do not have a bounded inverse. We can also see the unboundedness of this operator directly as follows:

$$\|T^{-1}e_i\| = \left\| \sum_{j=1}^{\infty} \lambda_j^{-1} \langle e_i, e_j \rangle e_j \right\| = \|\lambda_i^{-1} e_i\| = \lambda_i^{-1} \to \infty,$$

as $i \to \infty$.

9 A numerical illustration

Here we study the convolution operator, $K : L^2([0, 1]) \to L^2([0, 1])$ given by

$$(Ku)(x) = \int_{0}^{1} k(x - y)u(y)dy, \quad k(x) = \frac{1}{\sqrt{2\pi \gamma}} \exp\left(-\frac{x^2}{2\gamma^2}\right), \quad \gamma = 0.03.$$
This is a positive selfadjoint compact linear operator (compactness of this operator follows from Lemma 3.1). Applying the compact operator $K$ to a function in $L^2([0, 1])$ has a smoothing effect. This is illustrated in Figure 1, where we show the effect of applying $K$ on three different functions.

To study the spectral properties of $K$, we compute its eigenvalues and eigenvectors. To this end, we need to solve the following eigenvalue problem: find $\lambda_i \in \mathbb{R}$ and $e_i \in L^2(D)$ that satisfy

$$\int_0^1 k(x - y)e_k(y)dy = \lambda_k e_k(x), \quad \int_0^1 e_k^2(x) dx = 1,$$

for $i = 1, 2, 3, \ldots$. This problem can be solved numerically by discretizing the integral operator via quadrature:

$$\sum_{j=1}^n w_j k(x_i - x_j)e_k(x_j) = \lambda_k e_k(x_i), \quad i = 1, \ldots, n,$$

where $\{(x_i, w_i)\}_{i=1}^n$ are nodes and weights of an appropriate quadrature formula (below, we use composite midpoint rule). Letting $e_k^i = e_k(x_i)$, $K_{ij} = \frac{1}{\sqrt{2\pi\gamma}}k(x_i - x_j)$, and defining the matrix $W = \text{diag}(w_1, w_2, \ldots, w_n)$. The discretized eigenvalue problem is given by,

$$KWe_k = \lambda_k e_k.$$

This can be rewritten as the symmetric eigenvalue problem,

$$W^{1/2}KW^{1/2}\tilde{e}_k = \lambda_k \tilde{e}_k,$$

with $\tilde{e}_k = W^{1/2}e_k$. Notice upon solving the reformulated symmetric eigenvalue problem, we obtain eigenvalues $\lambda_k$ and eigenvectors $\tilde{e}_k$, with $\tilde{e}_k^T \tilde{e}_k = 1$. We can then obtain the eigenvectors $e_k$ using $e_k = W^{-1/2} \tilde{e}_k$. Also note that,

$$e_k^T We_k = 1.$$

This method of solving for eigenvalues and eigenvectors of the covariance operators is known as the Nyström’s method.

Here we discretize the eigenvalue problem using an $n$-point composite mid-point rule. The discretized operator $K$ becomes:

$$K_{ij} = \frac{1}{\sqrt{2\pi\gamma}} \exp \left( \frac{-((i - j)h)^2}{2\gamma^2} \right), \quad h = 1/n, \quad i, j \in \{1, \ldots, n\}.$$ 

In the results reported below, we used $n = 1024$. Figure 2 shows the eigenvalues of $K$, and Figure 3 shows some of the eigenvectors, obtained by solving the discretized eigenvalue problem. Notice that the higher order eigenvectors (eigenfunctions) are highly oscillatory.

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Figure 2: Eigenvalues of the convolution operator $K$.

Figure 3: A few of the eigenvectors of the convolution operator $K$.


