

## ANALYSIS OF THE HESSIAN FOR INVERSE SCATTERING PROBLEMS. PART III: INVERSE MEDIUM SCATTERING OF ELECTROMAGNETIC WAVES IN THREE DIMENSIONS

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**ABSTRACT.** Continuing our previous work [6, Inverse Problems, 2012, 28, 055002] and [5, Inverse Problems, 2012, 28, 055001], we address the ill-posedness of the inverse scattering problem of electromagnetic waves due to an inhomogeneous medium by studying the Hessian of the data misfit. We derive and analyze the Hessian in both Hölder and Sobolev spaces. Using an integral equation approach based on Newton potential theory and compact embeddings in Hölder and Sobolev spaces, we show that the Hessian can be decomposed into three components, all of which are shown to be compact operators. The implication of the compactness of the Hessian is that for small data noise and model error, the discrete Hessian can be approximated by a low-rank matrix. This in turn enables fast solution of an appropriately regularized inverse problem, as well as Gaussian-based quantification of uncertainty in the estimated inhomogeneity.

**1. Introduction.** A feature of many ill-posed inverse problems is that the Hessian operator of the data misfit functional is a compact operator with rapidly decaying eigenvalues. This is a manifestation of the typically sparse observations, which are informative about a limited number of modes of the infinite dimensional field we seek to infer. The Hessian operator (and its finite dimensional discretization) plays an important role in the analysis and solution of the inverse problem. In particular, the spectrum of the Hessian at the solution of the inverse problem determines the degree of ill-posedness and provides intuition on the construction of appropriate

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regularization strategies. This has been observed, analyzed, and exploited in several applications including shape optimization [13, 12] and inverse wave propagation [11, 14, 4], to name a few.

Moreover, solution of the inverse problem by the gold standard iterative method—Newton’s method—requires “inversion” of the Hessian at each iteration. Compactness of the Hessian of the data misfit functional accompanied by sufficiently fast eigenvalue decay permits a low rank approximation, which in turn facilitates rapid inversion or preconditioning of the regularized Hessian [11, 8]. Alternatively, solution of the linear system arising at each Newton iteration by a conjugate gradient method can be very fast if the data misfit Hessian is compact with rapidly decaying eigenvalues and the conjugate gradient iteration is preconditioned by the regularization operator [1]. Finally, under a Gaussian approximation to the Bayesian solution of the inverse problem, the covariance of the posterior probability distribution is given by the inverse of the Hessian of the negative log likelihood function. For Gaussian data noise and model error, this Hessian is given by an appropriately weighted Hessian of the data misfit operator, e.g., [20]. Here again, exploiting the low-rank character of the data misfit component of the Hessian is critical for rapidly approximating its inverse, and hence the uncertainty in the inverse solution [14, 4, 17, 7].

In all of the cases described above, compactness of the data misfit Hessian is a critical feature that enables fast solution of the inverse problem, scalability of solvers to high dimensions, and estimation of uncertainty in the solution. With this motivation, here we analyze the Hessian operator for inverse medium electromagnetic scattering problems, and study its compactness. This is a continuation of our previous efforts in analyzing the ill-posedness, via the compactness of the Hessian, of inverse shape acoustic scattering [5] and inverse medium acoustic scattering [6] problems. Our analysis is based on an integral equation formulation of the time harmonic Maxwell equations, adjoint methods, and compact embeddings in Hölder and Sobolev spaces. These tools allow us to analyze the Hessian of the data misfit in detail. The ideas of proof are similar to those in [6], but the details are more technical due to the complexity of the Maxwell equations. Numerical results demonstrating the theoretical results are not carried out in this paper since similar numerical validations can be found in our previous work [5, 6].

The remainder of the paper is organized as follows. Section 2 briefly derives and formulates forward and inverse electromagnetic scattering problems due to bounded inhomogeneity. We then derive the Hessian for the inverse problem in Section 3. Section 4 justifies the Hessian derivation by studying the well-posedness of the (incremental) forward and (incremental) adjoint equations, and the regularity of their solutions with respect to the smoothness of the inhomogeneity. Next, we analyze the Hessian in Hölder spaces in Section 5, and then extend the analysis to Sobolev spaces in Section 6. Finally, the conclusions of the paper are presented in Section 7.

## 2. Forward and inverse medium problems for electromagnetic scattering.

In this section, we briefly discuss forward electromagnetic scattering problems due to bounded inhomogeneity and the corresponding inverse problems. We begin by assuming that the time harmonic incident wave  $(\mathbf{E}^{ic}, \mathbf{H}^{ic})$  satisfies the Maxwell

equation

$$\begin{aligned} \nabla \times \mathbf{E}^{ic} - ik\mathbf{H}^{ic} &= \mathbf{0}, && \text{in } \mathbb{R}^3, \\ \nabla \times \mathbf{H}^{ic} + ik\mathbf{E}^{ic} &= \mathbf{0}, && \text{in } \mathbb{R}^3, \end{aligned}$$

where  $k > 0$  is the wave number,  $i^2 = -1$  the imaginary unit. The total field formulation for the scattering due to dielectric medium [10] can be equivalently cast into the following scattered field formulation:

$$\begin{aligned} (1a) \quad & \nabla \times \mathbf{E} - ik\mathbf{H} = \mathbf{0}, && \text{in } \mathbb{R}^3, \\ (1b) \quad & \nabla \times \mathbf{H} + ikn\mathbf{E} = ik(1 - n)\mathbf{E}^{ic}, && \text{in } \mathbb{R}^3, \\ (1c) \quad & \lim_{r \rightarrow \infty} (\mathbf{H} \times \mathbf{x} - r\mathbf{E}) = \mathbf{0}, && r = \|\mathbf{x}\|, \end{aligned}$$

where  $n \equiv n(\mathbf{x}) > 0$  is the distributed refractive index which is assumed to be 1 for the free space (if not, we can always normalize (1a)–(1b) to fulfill this assumption), and  $(\mathbf{E}, \mathbf{H})$  the scattered field. The Silver–Müller radiation condition (1c) is assumed to be valid uniformly in all directions  $\frac{\mathbf{x}}{\|\mathbf{x}\|}$  with  $\mathbf{x} \in \mathbb{R}^3$  denoting the vector of spatial coordinates. Note that it can also be written as  $\lim_{r \rightarrow \infty} r(\mathbf{H} \times \mathbf{n} - \mathbf{E}) = \mathbf{0}$ , where  $\mathbf{n}$  is the unit outward normal vector of the sphere with radius  $r$ . To the rest of the paper, the medium is assumed to be bounded, that is, there exists a bounded domain  $\Omega$  such that  $n(\mathbf{x}) = 1, \forall \|\mathbf{x}\| \in \mathbb{R}^3 \setminus \Omega$ . In other words,  $q = 1 - n$  has compact support in  $\Omega$ . In fact, we shall assume  $n \in C^{m,\alpha}(\mathbb{R}^3)$ , and hence  $q \in C_0^{m,\alpha}(\Omega)$ , throughout the paper, where  $m \in \mathbb{N}$ .

For the forward problem,  $n$  is given and we solve the forward equations (1a)–(1c) for the scattered field  $(\mathbf{E}, \mathbf{H})$ . For the inverse problem, on the other hand, given observation data  $(\mathbf{E}^{obs}, \mathbf{H}^{obs})$  over some compact subset  $\Omega^{obs} \subset \mathbb{R}^3$ , we are asked to infer the distribution of the refractive index  $n$ . One way to solve the inverse problem is to cast it into the following PDE-constrained optimization problem:

$$(2) \quad \min_q \mathcal{J} := \int_{\mathbb{R}^3} K(\mathbf{x}) \left( |\mathbf{E} - \mathbf{E}^{obs}|^2 + |\mathbf{H} - \mathbf{H}^{obs}|^2 \right) d\mathbf{x},$$

subject to the forward equations (1a)–(1c). Here,  $K(\mathbf{x})$  is the observation operator whose support is  $\Omega^{obs}$ . Note that we could have different observation operators for  $\mathbf{E}$  and  $\mathbf{H}$  but, for simplicity of the exposition, we assume they are the same and equal to  $K(\mathbf{x})$ . In order to cover several interesting observation operators,  $\Omega^{obs}$  is allowed to be quite general in this paper. In particular, it could be a closed subset in  $\mathbb{R}^3$  or a relative closed subset of a manifold in  $\mathbb{R}^3$ . For example,  $\Omega^{obs}$  could be a closed arc, or a closed curved, or a closed subset of a two dimensional manifold, or some two dimensional manifold. For convenience, we identify

$$\mathcal{K}\varphi := \int_{\mathbb{R}^3} K\varphi d\mathbf{x} = \int_{\Omega^{obs}} \varphi(\mathbf{y}) d\mathbf{y}.$$

We also permit pointwise observations in our analysis, i.e.  $\Omega^{obs} \equiv \{\mathbf{x}_j^{obs}\}_{j=1}^{N^{obs}}$ , and in this case we identify

$$(3) \quad \mathcal{K}\varphi := \int_{\mathbb{R}^3} K\varphi d\mathbf{x} = \sum_{j=1}^{N^{obs}} \varphi(\mathbf{x}_j^{obs}).$$

**3. Derivation of the Hessian.** In this section, we derive the gradient and Hessian of the data misfit (2) using a reduced space approach<sup>1</sup>, and the justification for our derivations is provided in Section 4. We begin with a useful observation on the radiation condition. Since the radiation condition (1c) is valid uniformly in all directions  $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ , we rewrite the radiation condition as

$$\mathbf{H} \times \mathbf{n} - \mathbf{E} = \varphi(r) = o(r^{-1}),$$

where  $r$  is the radius of a sufficiently large circle  $\Gamma_\infty$ , and  $\varphi(r) = o(r^{-1})$  means

$$\lim_{r \rightarrow \infty} r\varphi(r) = \mathbf{0}.$$

It can be seen that the cost functional (2) is real-valued while the constraints (1a)–(1c) are complex-valued. Consequently, the usual Lagrangian approach will not make sense and care must be taken. Following Kreutz-Delgado [16], we define the Lagrangian as

$$\begin{aligned} \mathcal{L} = & \mathcal{J} - \int_{\mathbb{R}^3} \overline{\mathbf{h}} \cdot (\nabla \times \mathbf{E} - ik\mathbf{H}) \, d\mathbf{x} + \int_{\mathbb{R}^3} \overline{\mathbf{e}} \cdot [\nabla \times \mathbf{H} + ikn\mathbf{E} - ik(1-n)\mathbf{E}^{ic}] \, d\mathbf{x} \\ & + \int_{\Gamma_\infty} \overline{\mathbf{e}}_r \cdot [\mathbf{H} \times \mathbf{n} - \mathbf{E} - \varphi(r)] \, ds \\ & - \int_{\mathbb{R}^3} \mathbf{h} \cdot (\nabla \times \overline{\mathbf{E}} + ik\overline{\mathbf{H}}) \, d\mathbf{x} + \int_{\mathbb{R}^3} \mathbf{e} \cdot [\nabla \times \overline{\mathbf{H}} - ikn\overline{\mathbf{E}} + ik(1-n)\overline{\mathbf{E}}^{ic}] \, d\mathbf{x} \\ & + \int_{\Gamma_\infty} \mathbf{e}_r \cdot [\overline{\mathbf{H}} \times \mathbf{n} - \overline{\mathbf{E}} - \overline{\varphi}(r)] \, ds \end{aligned}$$

where the overline, when acting on forward and adjoint states (and their variations), denotes the complex conjugate.

Taking the first variation of the Lagrangian with respect to  $\mathbf{e}, \mathbf{h}, \mathbf{e}_r$  in the directions  $\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\mathbf{e}}_r$  and arguing that the variations  $\hat{\mathbf{e}}, \hat{\mathbf{h}}, \hat{\mathbf{e}}_r$  are arbitrary yield the forward equations (1a)–(1c).

Now taking the first variation of the Lagrangian with respect to  $\mathbf{E}, \mathbf{H}$  in the directions  $\hat{\mathbf{E}}, \hat{\mathbf{H}}$ , using the following vector calculus identities

$$(4a) \quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

$$(4b) \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}),$$

and arguing that the variations  $\hat{\mathbf{E}}, \hat{\mathbf{H}}$  are arbitrary yield the following adjoint equations:

$$(5a) \quad \nabla \times \mathbf{e} - ik\mathbf{h} = -K(\mathbf{H} - \mathbf{H}^{obs}), \quad \text{in } \mathbb{R}^3,$$

$$(5b) \quad \nabla \times \mathbf{h} + ikn\mathbf{e} = K(\mathbf{E} - \mathbf{E}^{obs}), \quad \text{in } \mathbb{R}^3,$$

$$(5c) \quad \lim_{r \rightarrow \infty} (\mathbf{h} \times \mathbf{x} + r\mathbf{e}) = \mathbf{0}, \quad r = \|\mathbf{x}\|,$$

and

$$\mathbf{e}_r = -\mathbf{h} \times \mathbf{n} \quad \text{on } \Gamma_\infty.$$

It should be pointed out that the adjoint equations (5a)–(5c) are very similar to the forward equations (1a)–(1c) except for the minus sign in the radiation condition. This can be understood that the adjoint waves originate from infinity.

<sup>1</sup>See [19] for the difference between the full space and the reduced space approaches.

If we eliminate  $\mathbf{e}_r$ , the Lagrangian now becomes

$$\begin{aligned} \mathcal{L} = & \mathcal{J} - \int_{\mathbb{R}^3} \bar{\mathbf{h}} \cdot (\nabla \times \mathbf{E} - ik\mathbf{H}) \, d\mathbf{x} + \int_{\mathbb{R}^3} \bar{\mathbf{e}} \cdot [\nabla \times \mathbf{H} + ikn\mathbf{E} - ik(1-n)\mathbf{E}^{ic}] \, d\mathbf{x} \\ & - \int_{\Gamma_\infty} (\bar{\mathbf{h}} \times \mathbf{n}) \cdot [\mathbf{H} \times \mathbf{n} - \mathbf{E} - \varphi(r)] \, ds \\ & - \int_{\mathbb{R}^3} \mathbf{h} \cdot (\nabla \times \bar{\mathbf{E}} + ik\bar{\mathbf{H}}) \, d\mathbf{x} + \int_{\mathbb{R}^3} \mathbf{e} \cdot [\nabla \times \bar{\mathbf{H}} - ikn\bar{\mathbf{E}} + ik(1-n)\bar{\mathbf{E}}^{ic}] \, d\mathbf{x} \\ & - \int_{\Gamma_\infty} (\mathbf{h} \times \mathbf{n}) \cdot [\bar{\mathbf{H}} \times \mathbf{n} - \bar{\mathbf{E}} - \bar{\varphi}(r)] \, ds \end{aligned}$$

The gradient of the cost function acting in the direction  $\hat{q}$  is simply the variation of the Lagrangian with respect to  $q$  in the direction  $\hat{q}$ , i.e.,

$$(6) \quad D\mathcal{J}(q; \hat{q}) = ik \int_{\mathbb{R}^3} \left[ \mathbf{e} \cdot (\bar{\mathbf{E}} + \bar{\mathbf{E}}^{ic}) - \bar{\mathbf{e}} \cdot (\mathbf{E} + \mathbf{E}^{ic}) \right] \hat{q} \, d\mathbf{x},$$

which is clearly a real number.

For the sake of convenience in deriving the Hessian, the forward and adjoint equations are best expressed in the weak form. As a direct consequence of the above variational calculus steps, the forward equation in the weak form reads

$$\begin{aligned} \mathcal{S}(q, \mathbf{E}, \mathbf{H}) = & - \int_{\mathbb{R}^3} \bar{\mathbf{h}} \cdot (\nabla \times \mathbf{E} - ik\mathbf{H}) \, d\mathbf{x} \\ & + \int_{\mathbb{R}^3} \bar{\mathbf{e}} \cdot [\nabla \times \mathbf{H} + ikn\mathbf{E} - ik(1-n)\mathbf{E}^{ic}] \, d\mathbf{x} \\ (7) \quad & - \int_{\Gamma_\infty} (\bar{\mathbf{h}} \times \mathbf{n}) \cdot [\mathbf{H} \times \mathbf{n} - \mathbf{E} - \varphi(r)] \, ds = 0, \quad \forall (\hat{\mathbf{e}}, \hat{\mathbf{h}}). \end{aligned}$$

Similarly, the adjoint equation in the weak form is given by

$$\begin{aligned} \mathcal{A}(q, \mathbf{E}, \mathbf{H}, \mathbf{e}, \mathbf{h}) = & \int_{\mathbb{R}^3} \widehat{\mathbf{H}} \cdot [\nabla \times \mathbf{e} - ik\mathbf{h} + K(\mathbf{H} - \mathbf{H}^{obs})] \, d\mathbf{x} \\ & - \int_{\mathbb{R}^3} \widehat{\mathbf{E}} \cdot [\nabla \times \mathbf{h} + ikne - K(\mathbf{E} - \mathbf{E}^{obs})] \, d\mathbf{x} \\ (8) \quad & - \int_{\Gamma_\infty} (\widehat{\mathbf{H}} \times \mathbf{n}) \cdot (\mathbf{h} \times \mathbf{n} + \mathbf{e}) \, ds = 0, \quad \forall (\widehat{\mathbf{E}}, \widehat{\mathbf{H}}). \end{aligned}$$

Next, the (reduced) Hessian acting in the directions  $\hat{q}$  and  $\tilde{q}$  is obtained by simply taking the first variation of the gradient  $D\mathcal{J}(q, \hat{q})$  with respect to  $q, \mathbf{E}, \mathbf{H}, \mathbf{e}$  and  $\mathbf{h}$  in the directions  $\tilde{q}, \tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{e}}$  and  $\tilde{\mathbf{h}}$ , i.e.,

$$(9) \quad D^2\mathcal{J}(q; \hat{q}, \tilde{q}) = ik \int_{\mathbb{R}^3} \left[ \tilde{\mathbf{e}} \cdot (\bar{\mathbf{E}} + \bar{\mathbf{E}}^{ic}) - \bar{\tilde{\mathbf{e}}} \cdot (\mathbf{E} + \mathbf{E}^{ic}) + \mathbf{e} \cdot \bar{\tilde{\mathbf{E}}} - \bar{\mathbf{e}} \cdot \tilde{\mathbf{E}} \right] \hat{q} \, d\mathbf{x}$$

As mentioned at the beginning of this section, the reduced space approach is employed, and hence the variations  $\tilde{\mathbf{E}}$  and  $\tilde{\mathbf{e}}$  cannot be arbitrary. In fact, they are only admissible if the forward and adjoint equations are satisfied. As a direct consequence, the first variations of  $\mathcal{S}(q, \mathbf{E}, \mathbf{H})$  and  $\mathcal{A}(q, \mathbf{E}, \mathbf{H}, \mathbf{e}, \mathbf{h})$  must vanish, that

is,  $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$  is the solution of the following incremental forward equation:

$$(10) \quad \begin{aligned} & - \int_{\mathbb{R}^3} \overline{\tilde{\mathbf{h}}} \cdot (\nabla \times \tilde{\mathbf{E}} - ik\tilde{\mathbf{H}}) \, d\mathbf{x} + \int_{\mathbb{R}^3} \tilde{\mathbf{e}} \cdot [\nabla \times \tilde{\mathbf{H}} + ikn\tilde{\mathbf{E}} - ik\tilde{q}(\mathbf{E} + \mathbf{E}^{ic})] \, d\mathbf{x} \\ & - \int_{\Gamma_\infty} (\overline{\tilde{\mathbf{h}}} \times \mathbf{n}) \cdot (\tilde{\mathbf{H}} \times \mathbf{n} - \tilde{\mathbf{E}}) \, ds = 0, \quad \forall (\hat{\mathbf{e}}, \hat{\mathbf{h}}), \end{aligned}$$

and  $(\tilde{\mathbf{e}}, \tilde{\mathbf{h}})$  is the solution of the following incremental adjoint equation:

$$(11) \quad \begin{aligned} & \int_{\mathbb{R}^3} \overline{\tilde{\mathbf{H}}} \cdot (\nabla \times \tilde{\mathbf{e}} - ik\tilde{\mathbf{h}} + K\tilde{\mathbf{H}}) \, d\mathbf{x} - \int_{\mathbb{R}^3} \tilde{\mathbf{E}} \cdot (\nabla \times \tilde{\mathbf{h}} + ikn\tilde{\mathbf{e}} - K\tilde{\mathbf{E}} - ik\tilde{q}\mathbf{e}) \, d\mathbf{x} \\ & - \int_{\Gamma_\infty} (\overline{\tilde{\mathbf{H}}} \times \mathbf{n}) \cdot (\tilde{\mathbf{h}} \times \mathbf{n} + \tilde{\mathbf{e}}) \, ds = 0, \quad \forall (\hat{\mathbf{E}}, \hat{\mathbf{H}}). \end{aligned}$$

Consequently, the corresponding strong form of the incremental forward equation is

$$(12a) \quad \nabla \times \tilde{\mathbf{E}} - ik\tilde{\mathbf{H}} = \mathbf{0}, \quad \text{in } \mathbb{R}^3,$$

$$(12b) \quad \nabla \times \tilde{\mathbf{H}} + ikn\tilde{\mathbf{E}} = ik\tilde{q}(\mathbf{E} + \mathbf{E}^{ic}), \quad \text{in } \mathbb{R}^3,$$

$$(12c) \quad \lim_{r \rightarrow \infty} (\tilde{\mathbf{H}} \times \mathbf{x} - r\tilde{\mathbf{E}}) = \mathbf{0}, \quad r = \|\mathbf{x}\|,$$

and that of the incremental adjoint equation reads

$$(13a) \quad \nabla \times \tilde{\mathbf{e}} - ik\tilde{\mathbf{h}} = -K\tilde{\mathbf{H}}, \quad \text{in } \mathbb{R}^3,$$

$$(13b) \quad \nabla \times \tilde{\mathbf{h}} + ikn\tilde{\mathbf{e}} = K\tilde{\mathbf{E}} + ik\tilde{q}\mathbf{e}, \quad \text{in } \mathbb{R}^3,$$

$$(13c) \quad \lim_{r \rightarrow \infty} (\tilde{\mathbf{h}} \times \mathbf{x} + r\tilde{\mathbf{e}}) = \mathbf{0}, \quad r = \|\mathbf{x}\|.$$

It should be pointed out that the incremental forward pair  $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$  in (12a)–(12c) is a function of  $\tilde{q}$ , i.e.,  $(\tilde{\mathbf{E}}(\tilde{q}), \tilde{\mathbf{H}}(\tilde{q}))$  for a given variation  $\tilde{q}$ . Similar observation applies to the incremental adjoint pair  $(\tilde{\mathbf{e}}(\tilde{q}), \tilde{\mathbf{h}}(\tilde{q}))$  in (13a)–(13c). To avoid possible confusion, if  $\tilde{\mathbf{E}}$  is evaluated at  $\hat{q}$ , for example, we will write this dependency explicitly as  $\tilde{\mathbf{E}}(\hat{q})$ ; otherwise  $\tilde{\mathbf{E}}$  means  $\tilde{\mathbf{E}}(\tilde{q})$  implicitly.

Next, we need to convert the Hessian in (9) into a symmetric form that is convenient for our later analysis. The first step is to replace  $\tilde{q}$  by  $\hat{q}$  and choose  $(\hat{\mathbf{e}}, \hat{\mathbf{h}}) = (\tilde{\mathbf{e}}(\hat{q}), \tilde{\mathbf{h}}(\hat{q}))$  in the incremental forward equation (10). In the second step, we take  $(\hat{\mathbf{E}}, \hat{\mathbf{H}}) = (\tilde{\mathbf{E}}(\hat{q}), \tilde{\mathbf{H}}(\hat{q}))$  in the incremental adjoint equation (11). The last step is to subtract the resulting incremental forward equation from the complex conjugate of the resulting incremental adjoint equation. After some simple integration by parts using (4a) and cancellations, we obtain

$$\begin{aligned} ik \int_{\mathbb{R}^3} [\hat{\mathbf{e}}(\hat{q}) \cdot (\overline{\mathbf{E}} + \overline{\mathbf{E}^{ic}}) - \overline{\hat{\mathbf{e}}(\hat{q})} \cdot (\mathbf{E} + \mathbf{E}^{ic})] \hat{q} \, d\mathbf{x} &= ik \int_{\mathbb{R}^3} [\mathbf{e} \cdot \overline{\tilde{\mathbf{E}}(\hat{q})} - \overline{\mathbf{e}} \cdot \tilde{\mathbf{E}}(\hat{q})] \hat{q} \, d\mathbf{x} \\ &+ \int_{\mathbb{R}^3} K [\overline{\tilde{\mathbf{H}}(\hat{q})} \cdot \tilde{\mathbf{H}}(\hat{q}) + \overline{\tilde{\mathbf{H}}(\hat{q})} \cdot \tilde{\mathbf{H}}(\hat{q})] \, d\mathbf{x} + \int_{\mathbb{R}^3} K [\overline{\tilde{\mathbf{E}}(\hat{q})} \cdot \tilde{\mathbf{E}}(\hat{q}) + \overline{\tilde{\mathbf{E}}(\hat{q})} \cdot \tilde{\mathbf{E}}(\hat{q})] \, d\mathbf{x} \end{aligned}$$

which, after combining with (9), gives the desired symmetric form of the Hessian as

$$\begin{aligned}
 D^2 \mathcal{J}(q; \hat{q}, \tilde{q}) &= \underbrace{\int_{\mathbb{R}^3} K \left[ \tilde{\mathbf{E}}(\hat{q}) \cdot \overline{\tilde{\mathbf{E}}(\tilde{q})} + \overline{\tilde{\mathbf{E}}(\hat{q})} \cdot \tilde{\mathbf{E}}(\tilde{q}) \right] dx}_{\mathcal{H}_1(q; \hat{q}, \tilde{q})} \\
 (14) \quad &+ \underbrace{\int_{\mathbb{R}^3} K \left[ \tilde{\mathbf{H}}(\hat{q}) \cdot \overline{\tilde{\mathbf{H}}(\tilde{q})} + \overline{\tilde{\mathbf{H}}(\hat{q})} \cdot \tilde{\mathbf{H}}(\tilde{q}) \right] dx}_{\mathcal{H}_2(q; \hat{q}, \tilde{q})} \\
 &\underbrace{ik \int_{\mathbb{R}^3} \left[ \mathbf{e} \cdot \overline{\tilde{\mathbf{E}}(\tilde{q})} - \overline{\mathbf{e}} \cdot \tilde{\mathbf{E}}(\tilde{q}) \right] \hat{q} dx + ik \int_{\mathbb{R}^3} \left[ \mathbf{e} \cdot \overline{\tilde{\mathbf{E}}(\hat{q})} - \overline{\mathbf{e}} \cdot \tilde{\mathbf{E}}(\hat{q}) \right] \tilde{q} dx}_{\mathcal{H}_3(q; \hat{q}, \tilde{q})}.
 \end{aligned}$$

**4. Regularity of the forward and adjoint solutions.** In this section we are going to justify what we have done in Section 3 by studying the regularity of the (incremental) forward and (incremental) adjoint solutions with respect to the medium  $q$  and its variations  $\hat{q}, \tilde{q}$ . In particular, for sufficiently smooth inhomogeneity, the solutions turn out to be classical by using an integral equation method, as we shall show. As a by-product, we shall also show that the (incremental) forward and (incremental) adjoint equations are well-posed.

Let us introduce the following standard volume potentials (also known as Newton potentials) [15, 10]:

$$(15) \quad w(\mathbf{x}) = T\varphi(\mathbf{x}) = \int_{\mathbb{R}^3} \Phi(\mathbf{x}, \mathbf{y})\varphi(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^3,$$

where  $\Phi$  is either the fundamental solution of the (incremental) forward equation(s) defined as

$$\Phi(\mathbf{x}, \mathbf{y}) = \frac{e^{ik\|\mathbf{x}-\mathbf{y}\|}}{4\pi \|\mathbf{x} - \mathbf{y}\|},$$

or the fundamental solution of the (incremental) adjoint solution(s):

$$\Phi(\mathbf{x}, \mathbf{y}) = \frac{e^{-ik\|\mathbf{x}-\mathbf{y}\|}}{4\pi \|\mathbf{x} - \mathbf{y}\|}.$$

We recall the standard mapping properties of  $T$  defined in (15) here.

**Lemma 4.1.** *Let  $\varphi \in C_0^{m,\alpha}(\mathbb{R}^3) := C_0(\mathbb{R}^3) \cap C^{m,\alpha}(\mathbb{R}^3)$ , where  $\alpha \in (0, 1]$  and  $m \in \mathbb{N}$ . Then  $w \in C^{m+2,\alpha}(\Omega)$ , and  $\|w\|_{C^{m+2,\alpha}(\Omega)} \leq c \|\varphi\|_{C^{m,\alpha}(\Omega)}$ , where  $\text{supp}(\varphi) \subset \Omega \subseteq \mathbb{R}^3$ . Moreover, the orders of differentiation and integration can be interchanged.*

*Proof.* See [6, Lemma 1] for a proof. □

The following representation formula is an easy variant of the Stratton and Chu theorem [10, Theorem 6.1].

**Lemma 4.2.** *Let  $\mathcal{E}, \mathcal{H} \in C^1(\mathbb{R}^3)$ , there holds*

$$\begin{aligned}
 \mathcal{E}(\mathbf{x}) = & - \lim_{r \rightarrow \infty} \nabla \times \int_{\Gamma_\infty} \mathbf{n}(\mathbf{y}) \times \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, ds(\mathbf{y}) \\
 & + \lim_{r \rightarrow \infty} \nabla \int_{\Gamma_\infty} \mathbf{n}(\mathbf{y}) \cdot \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, ds(\mathbf{y}) \\
 & - \lim_{r \rightarrow \infty} ik \int_{\Gamma_\infty} \mathbf{n}(\mathbf{y}) \times \mathcal{H}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, ds(\mathbf{y}) \\
 & + \nabla \times \int_{\mathbb{R}^3} [\nabla \times \mathcal{E}(\mathbf{y}) - ik\mathcal{H}(\mathbf{y})] \Phi(\mathbf{x}, \mathbf{y}) \, dy \\
 & - \nabla \int_{\mathbb{R}^3} \nabla \cdot \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, dy \\
 (16) \quad & + ik \int_{\mathbb{R}^3} [\nabla \times \mathcal{H}(\mathbf{y}) + ik\mathcal{E}(\mathbf{y})] \Phi(\mathbf{x}, \mathbf{y}) \, dy, \quad \forall \mathbf{x} \in \mathbb{R}^3.
 \end{aligned}$$

Next, we show that the boundary terms in (16) vanish for a large class of problems.

**Lemma 4.3.** *Let  $\mathcal{E}, \mathcal{H} \in C^1(\mathbb{R}^3)$  satisfy*

$$(17a) \quad \nabla \times \mathcal{E} - ik\mathcal{H} = \mathbf{f}, \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},$$

$$(17b) \quad \nabla \times \mathcal{H} + ik\mathcal{E} = \mathbf{g}, \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},$$

$$(17c) \quad \lim_{r \rightarrow \infty} (\mathcal{H} \times \mathbf{x} \pm r\mathcal{E}) = \mathbf{0}, \quad r = \|\mathbf{x}\|,$$

$$(17d) \quad 2\Re \int_{\Gamma_\infty} \overline{\mathcal{H}} \cdot \mathbf{f} \, ds < \infty, \quad 2\Re \int_{\Gamma_\infty} \mathcal{E} \cdot \overline{\mathbf{g}} \, ds < \infty,$$

where  $\Re$  is an operator that returns the real part of its argument. Then

$$\lim_{r \rightarrow \infty} \int_{\Gamma_\infty} \|\mathcal{H} \times \mathbf{n}\|^2 \, ds = \mathcal{O}(1), \quad \text{and} \quad \lim_{r \rightarrow \infty} \int_{\Gamma_\infty} \|\mathcal{E}\|^2 \, ds = \mathcal{O}(1).$$

Moreover,

$$\begin{aligned}
 & - \lim_{r \rightarrow \infty} \nabla \times \int_{\Gamma_\infty} \mathbf{n}(\mathbf{y}) \times \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, ds(\mathbf{y}) \\
 & + \lim_{r \rightarrow \infty} \nabla \int_{\Gamma_\infty} \mathbf{n}(\mathbf{y}) \cdot \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, ds(\mathbf{y}) \\
 & - \lim_{r \rightarrow \infty} ik \int_{\Gamma_\infty} \mathbf{n}(\mathbf{y}) \times \mathcal{H}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, ds(\mathbf{y}) = \mathbf{0}, \quad \forall \mathbf{x} \in \mathbb{R}^3.
 \end{aligned}$$

*Proof.* First, from the assumption on the radiation condition (17c) we have

$$\int_{\Gamma_\infty} \|\mathcal{H} \times \mathbf{n} \pm \mathcal{E}\|^2 \, ds = 0,$$

from which it follows that

$$\int_{\Gamma_\infty} \left( \|\mathcal{H} \times \mathbf{n}\|^2 + \|\mathcal{E}\|^2 \right) \, ds = \mp 2\Re \int_{\Gamma_\infty} (\mathbf{n} \times \mathcal{E}) \cdot \overline{\mathcal{H}} \, ds.$$

Second, integrating both sides of (4a) on  $\mathbb{R}^3 \setminus \bar{\Omega}$  with  $\mathbf{A} = \mathcal{E}, \mathbf{B} = \overline{\mathcal{H}}$  and using Gauss divergence theorem together with assumptions (17a)–(17b) give

$$\int_{\Gamma_\infty} (\mathbf{n} \times \mathcal{E}) \cdot \overline{\mathcal{H}} \, ds + \int_{\partial\Omega} (\mathbf{n} \times \mathcal{E}) \cdot \overline{\mathcal{H}} \, ds = ik \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \left( \|\mathcal{H}\|^2 - \|\mathcal{E}\|^2 \right) \, dx - \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \mathcal{E} \cdot \bar{\mathbf{g}} \, dx + \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \overline{\mathcal{H}} \cdot \mathbf{f} \, dx.$$

It follows that

$$\begin{aligned} & \mp 2\Re \int_{\Gamma_\infty} (\mathbf{n} \times \mathcal{E}) \cdot \overline{\mathcal{H}} \, ds \\ &= \pm 2\Re \int_{\partial\Omega} (\mathbf{n} \times \mathcal{E}) \cdot \overline{\mathcal{H}} \, ds \pm 2\Re \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \mathcal{E} \cdot \bar{\mathbf{g}} \, dx \mp 2\Re \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \overline{\mathcal{H}} \cdot \mathbf{f} \, dx. \end{aligned}$$

Consequently, we have the identity

$$\begin{aligned} \int_{\Gamma_\infty} \left( \|\mathcal{H} \times \mathbf{n}\|^2 + \|\mathcal{E}\|^2 \right) \, ds &= \pm 2\Re \int_{\partial\Omega} (\mathbf{n} \times \mathcal{E}) \cdot \overline{\mathcal{H}} \, ds \\ &\quad \pm 2\Re \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \mathcal{E} \cdot \bar{\mathbf{g}} \, dx \mp 2\Re \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \overline{\mathcal{H}} \cdot \mathbf{f} \, dx, \end{aligned}$$

whose right side is finite, and so is the left side. The first assertion is therefore justified. The second assertion is the direct consequence of the first and the proof of [10, Theorem 6.6].  $\square$

Define, for any  $\mathbf{x} \in \Omega$ ,

$$\begin{aligned} (\mathcal{Q}[\tilde{q}]^{\mathcal{E}})(\mathbf{x}) &= \nabla \int_{\Omega} \frac{\tilde{q}(\mathbf{y})}{1 - q(\mathbf{y})} \nabla \cdot \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \\ \mathcal{R}(\mathcal{E}, \mathcal{H})(\mathbf{x}) &= -\nabla \times \int_{\Omega^{obs}} K(\mathbf{y}) \mathcal{H} \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \\ &\quad + \frac{i}{k} \nabla \nabla \cdot \int_{\Omega^{obs}} \frac{1}{1 - q(\mathbf{y})} K(\mathbf{y}) \mathcal{E} \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \\ &\quad + ik \int_{\Omega^{obs}} K(\mathbf{y}) \mathcal{E} \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \\ (\mathcal{T}[q]^{\mathcal{E}})(\mathbf{x}) &= \nabla \int_{\Omega} \frac{1}{1 - q(\mathbf{y})} \nabla q(\mathbf{y}) \cdot \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} + k^2 \int_{\Omega} q(\mathbf{y}) \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \\ (\hat{\mathcal{T}}[\tilde{q}]^{\mathcal{E}})(\mathbf{x}) &= \nabla \int_{\Omega} \frac{1}{1 - q(\mathbf{y})} \nabla \tilde{q}(\mathbf{y}) \cdot \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} + k^2 \int_{\Omega} \tilde{q}(\mathbf{y}) \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \end{aligned}$$

and  $\mathcal{I}$  as the identity operator. To clarify our notations, we note that, for example,  $\mathcal{Q}[\tilde{q}]$  is a linear operator that depends on  $\tilde{q}$ . Here is a direct consequence of Lemma 4.1.

**Corollary 1.** *Suppose  $q, \tilde{q} \in C_0^{m,\alpha}(\Omega)$ ,  $m \in \mathbb{N}$ .*

- i)  $\mathcal{Q}[\tilde{q}]$  maps  $C^{s+1,\beta}(\Omega)$  continuously into  $C^{s+1,\beta}(\Omega)$  for  $s + \beta \leq m + \alpha$ .
- ii) The cartesian components of  $\mathcal{R}(\mathcal{E}, \mathcal{H})$  are analytic.
- iii)  $\mathcal{T}[q], \hat{\mathcal{T}}[\tilde{q}]$  map  $C^{p-1,\alpha}(\Omega)$  continuously into  $C^{p,\alpha}(\Omega)$  for  $1 \leq p \leq m$ , and  $\mathcal{T}[q], \hat{\mathcal{T}}[\tilde{q}] : C^{p-1,\alpha}(\Omega) \rightarrow C^{s,\beta}(\Omega)$  are compact operators for  $p - 1 + \alpha \leq s + \beta < p + \alpha$ .

*Proof.* It is sufficient to prove the third assertion. A straightforward application of Lemma 4.1 shows that  $\mathcal{T}[q]$  is a continuous operator from  $C^{p-1,\alpha}(\Omega)$  to  $C^{p,\alpha}(\Omega)$ . Moreover, the compactness is trivial due to compact embeddings in Hölder spaces [21].  $\square$

We are in the position to discuss representation formulas for (incremental) forward and (incremental) adjoint equations.

**Proposition 1.** *Solutions of the (incremental) forward and (incremental) adjoint equations satisfy:*

$$\begin{aligned} (18) \quad & (\mathcal{I} + \mathcal{T}[q]) \mathbf{E} = -\mathcal{T}[q] \mathbf{E}^{ic}, \\ (19) \quad & (\mathcal{I} + \mathcal{T}[q]) \tilde{\mathbf{E}} = -\hat{\mathcal{T}}[\tilde{q}] (\mathbf{E} + \mathbf{E}^{ic}) - \mathcal{Q}[\tilde{q}] \mathbf{E}, \\ (20) \quad & (\mathcal{I} + \mathcal{T}[q]) \mathbf{e} = \mathcal{R} (\mathbf{E} - \mathbf{E}^{obs}, \mathbf{H} - \mathbf{H}^{obs}), \\ (21) \quad & (\mathcal{I} + \mathcal{T}[q]) \tilde{\mathbf{e}} = \mathcal{R} (\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) - \hat{\mathcal{T}}[\tilde{q}] \mathbf{e} - \mathcal{Q}[\tilde{q}] \mathbf{e}. \end{aligned}$$

Conversely, suppose that  $\mathbf{E}, \tilde{\mathbf{E}}, \mathbf{e}, \tilde{\mathbf{e}}$  are solutions of the integral equations (18)–(21), and define,  $\forall \mathbf{x} \in \mathbb{R}^3$ ,

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &:= -\mathcal{T}[q] \mathbf{E} - \mathcal{T}[q] \mathbf{E}^{ic}, \\ \tilde{\mathbf{E}}(\mathbf{x}) &:= -\mathcal{T}[q] \tilde{\mathbf{E}} - \hat{\mathcal{T}}[\tilde{q}] (\mathbf{E} + \mathbf{E}^{ic}) - \mathcal{Q}[\tilde{q}] \mathbf{E}, \\ \mathbf{e}(\mathbf{x}) &:= -\mathcal{T}[q] \mathbf{e} + \mathcal{R} (\mathbf{E} - \mathbf{E}^{obs}, \mathbf{H} - \mathbf{H}^{obs}), \\ \tilde{\mathbf{e}}(\mathbf{x}) &:= -\mathcal{T}[q] \tilde{\mathbf{e}} + \mathcal{R} (\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) - \hat{\mathcal{T}}[\tilde{q}] \mathbf{e} - \mathcal{Q}[\tilde{q}] \mathbf{e}, \end{aligned}$$

$\mathbf{H} := \nabla \times \mathbf{E} / (ik)$ ,  $\mathbf{h} := \nabla \times \mathbf{e} / (ik) + K (\mathbf{H} - \mathbf{H}^{obs}) / (ik)$ ,  $\tilde{\mathbf{H}} := \nabla \times \tilde{\mathbf{E}} / (ik)$ , and  $\tilde{\mathbf{h}} := \nabla \times \tilde{\mathbf{e}} / (ik) + K \tilde{\mathbf{H}} / (ik)$ . Then  $(\mathbf{E}, \mathbf{H}), (\tilde{\mathbf{E}}, \tilde{\mathbf{H}}), (\mathbf{e}, \mathbf{h}), (\tilde{\mathbf{e}}, \tilde{\mathbf{h}})$  are solutions of the (incremental) forward and (incremental) adjoint equations.

*Proof.* We first observe that the (incremental) forward and (incremental) adjoint equations satisfy Lemma 4.3 since  $q, \tilde{q}$  and  $K$  have compact support. Lemma 4.2 then shows that all these equations have the following common representation formula

$$\begin{aligned} \mathcal{E}(\mathbf{x}) &= \nabla \times \int_{\mathbb{R}^3} [\nabla \times \mathcal{E}(\mathbf{y}) - ik\mathcal{H}(\mathbf{y})] \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \\ &\quad - \nabla \int_{\mathbb{R}^3} \nabla \cdot \mathcal{E}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \\ (22) \quad &\quad + ik \int_{\mathbb{R}^3} [\nabla \times \mathcal{H}(\mathbf{y}) + ik\mathcal{E}(\mathbf{y})] \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \quad \forall \mathbf{x} \in \mathbb{R}^3. \end{aligned}$$

We provide the proof for only (21) since the others are simpler. From (13b) we find that

$$\nabla \cdot \tilde{\mathbf{e}} = -\frac{1}{n} \nabla n \cdot \tilde{\mathbf{e}} - \frac{i}{kn} \nabla \cdot (K \tilde{\mathbf{E}}) + \frac{1}{n} \nabla \cdot (\tilde{q} \mathbf{e}),$$

which, together with (13a)–(13b), can be substituted to (22) to end the proof of (21) by using the fact

$$-\frac{i}{k} \int_{\mathbb{R}^3} \frac{1}{n} \nabla \cdot (K \tilde{\mathbf{E}}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} = \frac{i}{k} \nabla \cdot \int_{\Omega^{obs}} \frac{K(\mathbf{y})}{n(\mathbf{y})} \tilde{\mathbf{E}}(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y},$$

owing to  $\nabla_{\mathbf{x}}\Phi(\mathbf{x}, \mathbf{y}) = -\nabla_{\mathbf{y}}\Phi(\mathbf{x}, \mathbf{y})$ ,  $\text{supp}q \cap \text{supp}K = \emptyset$ , and integration by parts. The converse is long and tedious, and hence omitted, though one can similarly follow [10, Theorem 9.2].  $\square$

Proposition 1 shows that we can equivalently study the existence, uniqueness, and stability of (incremental) forward and (incremental) adjoint solutions via the integral equations (18)–(21).

**Theorem 4.4.** *Let  $q, \tilde{q} \in C_0^{m,\alpha}(\Omega)$ ,  $m \in \mathbb{N}$ , there exists a unique solution to*

*i) the forward integral equation (18) such that  $\mathbf{E} \in C^{m,\alpha}(\Omega)$ ,  $\mathbf{H} \in C^{m+1,\alpha}(\Omega)$ , and*

$$\begin{aligned} \|\mathbf{E}\|_{C^{m,\alpha}(\Omega)} &\leq c \|q\|_{C^{m,\alpha}(\Omega)} \|\mathbf{E}^{ic}\|_{C^{m,\alpha}(\Omega)}, \\ \|\mathbf{H}\|_{C^{m+1,\alpha}(\Omega)} &\leq c \|q\|_{C^{m,\alpha}(\Omega)} \|\mathbf{E} + \mathbf{E}^{ic}\|_{C^{m,\alpha}(\Omega)}. \end{aligned}$$

*ii) to the incremental forward integral equation (19) such that  $\tilde{\mathbf{E}} \in C^{m,\alpha}(\Omega)$ ,  $\tilde{\mathbf{H}} \in C^{m+1,\alpha}(\Omega)$ , and  $\|\tilde{\mathbf{H}}\|_{C^{m+1,\alpha}(\Omega)} \leq c \|\tilde{\mathbf{E}}\|_{C^{m,\alpha}(\Omega)}$  where*

$$\|\tilde{\mathbf{E}}\|_{C^{m,\alpha}(\Omega)} \leq c \|\tilde{q}\|_{C^{m,\alpha}(\Omega)} \left( \|\mathbf{E}\|_{C^{m,\alpha}(\Omega)} + \|\mathbf{E}^{ic}\|_{C^{m,\alpha}(\Omega)} \right).$$

*iii) the adjoint integral equation (20) such that  $\mathbf{e} \in C^{m,\alpha}(\Omega)$  and  $\|\mathbf{e}\|_{C^{m,\alpha}(\Omega)} \leq c (\|\mathbf{E} - \mathbf{E}^{obs}\|_{\infty} + \|\mathbf{H} - \mathbf{H}^{obs}\|_{\infty})$ .*

*iv) the incremental adjoint integral equation (21) such that  $\tilde{\mathbf{e}} \in C^{m,\alpha}(\Omega)$  and  $\|\tilde{\mathbf{e}}\|_{C^{m,\alpha}(\Omega)} \leq c (\|\tilde{\mathbf{E}}\|_{\infty} + \|\tilde{\mathbf{H}}\|_{\infty} + \|\mathbf{e}\|_{C^{m,\alpha}(\Omega)})$ .*

*Proof.* We begin with the first assertion. Since  $\mathbf{E}^{ic}$  is analytic Corollary 1 shows that the right side of (18) belongs to  $C^{m,\alpha}(\Omega)$ , and that  $\mathcal{T}$  is compact in  $C^{m-1,\beta}(\Omega)$ . By the Riesz-Fredholm theory [9], we only need to prove the uniqueness of (18). That is, we need to prove that  $(\mathcal{I} + \mathcal{T}[q])\mathbf{E} = \mathbf{0}$  in  $C^{m-1,\alpha}(\Omega)$  implies  $\mathbf{E} = \mathbf{0}$  in  $C^{m-1,\alpha}(\Omega)$ . Now, Proposition 1 indicates that  $\mathbf{E}$  and  $\mathbf{H} = \nabla \times \mathbf{E}/(ik)$  is the solution of (1a)–(1c) with  $\mathbf{E}^{ic} = \mathbf{0}$ . Following Step 2 of the proof of Lemma 4.3 in  $\Omega$  with  $\mathcal{E} = \mathbf{E}$ ,  $\mathcal{H} = \mathbf{H}$ ,  $\mathbf{f} = \mathbf{g} = \mathbf{0}$ , we find that

$$2\Re \int_{\partial\Omega} (\mathbf{n} \times \mathcal{E}) \cdot \overline{\mathcal{H}} \, ds = 0,$$

which, by Steps 1 and 2 of the proof of Lemma 4.3, implies that

$$\lim_{r \rightarrow \infty} \int_{\Gamma_{\infty}} \|\mathbf{E}\|^2 \, ds = 0.$$

Now, since  $(\mathbf{E}, \mathbf{H})$  satisfies the Maxwell equation in  $\mathbb{R}^3 \setminus \overline{\Omega}$ , each component of  $\mathbf{E}$  satisfies the Helmholtz equation in  $\mathbb{R}^3 \setminus \overline{\Omega}$ , and we conclude that  $\mathbf{E} = \mathbf{H} = \mathbf{0}$  in  $\mathbb{R}^3 \setminus \overline{\Omega}$  by Rellich lemma [10]. A result in [10, Theorem 9.3] then shows that  $\mathbf{E} = \mathbf{H} = \mathbf{0}$  in  $\mathbb{R}^3$ . Invoking the Riesz-Fredholm theory yields that there exists a unique  $\mathbf{E} \in C^{m-1,\alpha}(\Omega)$  and

$$\|\mathbf{E}\|_{C^{m-1,\alpha}(\Omega)} \leq c \|\mathcal{T}[q]\mathbf{E}^{ic}\|_{C^{m-1,\alpha}(\Omega)} \leq c \|q\|_{C^{m,\alpha}(\Omega)} \|\mathbf{E}^{ic}\|_{C^{m-1,\alpha}(\Omega)},$$

where we have used [5, Lemma 5] in the last inequality. Unless otherwise stated, the constant  $c$  at different places may have different value. Using Lemma 4.1 shows that in fact  $\mathbf{E} \in C^{m,\alpha}(\Omega)$  and  $\|\mathbf{E}\|_{C^{m,\alpha}(\Omega)} \leq c \|\mathbf{E}\|_{C^{m-1,\alpha}(\Omega)}$ . Next, using Lemma

4.1 and the fact  $\nabla \times \nabla(\cdot) = 0$ , we see that  $\mathbf{H} \in C^{m+1,\alpha}(\Omega)$  and  $\|\mathbf{H}\|_{C^{m+1,\alpha}(\Omega)} \leq c\|q\|_{C^{m,\alpha}(\Omega)}\|\mathbf{E} + \mathbf{E}^{ic}\|_{C^{m,\alpha}(\Omega)}$ .

Similarly, for the incremental forward equation, we observe that the right side of (19) lives in  $C^{m,\alpha}(\Omega)$  by an application of Lemma 4.1. An application of the Riesz-Fredholm theory shows that  $\tilde{\mathbf{E}} \in C^{m-1,\alpha}(\Omega)$  with

$$\|\tilde{\mathbf{E}}\|_{C^{m-1,\alpha}(\Omega)} \leq c\|\tilde{q}\|_{C^{m,\alpha}(\Omega)} \left( \|\mathbf{E}\|_{C^{m,\alpha}(\Omega)} + \|\mathbf{E}^{ic}\|_{C^{m,\alpha}(\Omega)} \right),$$

and the property of the Newton potential in Lemma 4.1 improves the result to  $\tilde{E} \in C^{m,\alpha}(\Omega)$ .

The proofs for the adjoint and incremental adjoint equations follow similarly by observing that  $\mathcal{R}(\cdot, \cdot)$  is analytic due to the fact  $\Omega^{obs} \cap \Omega = \emptyset$ . □

**Remark 1.** Let us define  $\min\{(m, \alpha), (s + 1, \beta)\}$  to be equal to  $(m, \alpha)$  if  $m + \alpha \leq s + 1 + \beta$  and to  $(s + 1, \beta)$  otherwise. Then, a more general result can be obtained: for the forward equation, for example, if  $1 \leq p \leq m$ , for  $p - 1 + \alpha \leq s + \beta < p + \alpha$  there exists a unique solution to (18) such that  $\mathbf{E} \in C^{\min\{(m,\alpha),(s+1,\beta)\}}(\Omega)$ ,  $\mathbf{H} \in C^{\min\{(m+1,\alpha),(s+2,\beta)\}}(\Omega)$ . Moreover,  $\|\mathbf{E}\|_{C^{\min\{(m,\alpha),(s+1,\beta)\}}(\Omega)} \leq c\|\mathbf{E}^{ic}\|_{C^{m-1,\alpha}(\Omega)}$ , and  $\|\mathbf{H}\|_{C^{\min\{(m+1,\alpha),(s+2,\beta)\}}(\Omega)} \leq c\|\mathbf{E}^{ic}\|_{C^{m-1,\alpha}(\Omega)}$ . However, for clarity of the exposition, we consider only the case  $p = s = m, \beta = \alpha$ .

We are now in the position to justify our derivations of gradient and Hessian in Section 3.

**Theorem 4.5.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain. Assume that  $q, \hat{q}$  and  $\tilde{q}$  belong to  $C_0^{1,\alpha}(\Omega)$ . Then, the cost functional (2) is twice continuously Fréchet differentiable; and hence the gradient (6) and Hessian (14) are well-defined.*

*Proof.* First, observe that we have used the Gâteaux variation to derive the gradient and Hessian in Section 3. Now it is evident that both  $D\mathcal{J}(q; \hat{q})$  and  $D^2\mathcal{J}(q; \hat{q}, \tilde{q})$  are linear and continuous with respect to  $\hat{q}$  (and  $\tilde{q}$ ) by Theorem 4.4. Moreover, continuous dependence on  $q$  of  $\mathbf{E}, \mathbf{H}$  from Theorem 4.4 implies the continuous dependence on  $q$  of  $\mathbf{e}, \tilde{\mathbf{E}},$  and  $\tilde{\mathbf{H}}$ , which in turn implies the continuity of  $D\mathcal{J}(q; \hat{q})$  and  $D^2\mathcal{J}(q; \hat{q}, \tilde{q})$  with respect to  $q$ . Hence, a classical result on sufficiency for Fréchet derivative [3] ends the proof. □

**5. Analysis of the Hessian in Hölder spaces.** In this section we study the behavior of the Hessian at a fixed refractive distribution  $n$ , i.e.,  $q = 1 - n \in C_0^{m,\alpha}(\Omega)$ . Similar to our previous work [6] on medium inverse acoustic scattering, the Hessian of the inverse medium scattering problem for electromagnetic waves turns out to be compact for all  $q$  as we shall show. For concreteness, we restricted ourselves to two exemplary cases of the observation operator, namely, the observation is everywhere on a compact subset  $\Omega^{obs}$  having non-trivial  $r$ -dimensional Lebesgue measure for some  $1 \leq r \leq 3$  (we call this case as continuous observation) and pointwise observation  $\Omega^{obs} = \{\mathbf{x}_j^{obs}\}_{j=1}^{N^{obs}}$ .

From Theorem 4.4, one can define a solution operator for the incremental forward equation that maps  $\tilde{q} \in C_0^{m,\alpha}(\Omega)$  to the corresponding  $\tilde{\mathbf{E}}(\tilde{q}) \in C^{m-1,\alpha}(\Omega)$ . However, for the sake of simplicity in writing, we identify the incremental forward

solution  $\tilde{\mathbf{E}}$  with that solution operator as:

$$C_0^{m,\alpha}(\Omega) \ni \tilde{q} \mapsto \tilde{\mathbf{E}}(\tilde{q}) = -(\mathcal{I} + \mathcal{T}[q])^{-1} \hat{\mathcal{T}}[\tilde{q}] (\mathbf{E} + \mathbf{E}^{ic}) - (\mathcal{I} + \mathcal{T}[q])^{-1} \mathcal{Q}[\tilde{q}] \mathbf{E} \in C^{m-1,\alpha}(\Omega).$$

**Lemma 5.1.**  $\tilde{\mathbf{E}}(\tilde{q}) : C_0^{m,\alpha}(\Omega) \rightarrow C^{m-1,\alpha}(\Omega)$  is a compact operator.

*Proof.* First, owing to the Riesz-Fredholm theory,  $(I + \mathcal{T}[q])^{-1}$  is a bounded map from  $C^{m-1,\alpha}(\Omega)$  to  $C^{m-1,\alpha}(\Omega)$ . Second, due to  $\mathbf{E} \in C^{m,\alpha}(\Omega) \subset C^{m-1,\alpha}(\Omega)$ , Corollary 1 shows that both  $\hat{\mathcal{T}}$  and  $\mathcal{Q}$ , as a function of  $\tilde{q}$ , map  $C_0^{m,\alpha}(\Omega)$  continuously into  $C^{m,\alpha}(\Omega)$ , and hence compactly embedded into  $C^{m-1,\alpha}(\Omega)$  [21]. Finally, since  $\tilde{\mathbf{E}}$  is a sum of two operators, each of which is a composition of a continuous and a compact operators, it is compact.  $\square$

As a direct consequence,  $\tilde{\mathbf{E}}(\cdot)|_{\Omega^{obs}} : C_0^{m,\alpha}(\Omega) \rightarrow C^{m-1,\alpha}(\Omega^{obs})$  is still a compact operator since restricting to  $\Omega^{obs}$  is a continuous operation.

If the observation is continuous, the first Gauss-Newton part of the Hessian, namely  $\mathcal{H}_1(q; \hat{q}, \tilde{q})$ , can be now rewritten as

$$\begin{aligned} \mathcal{H}_1(q; \hat{q}, \tilde{q}) &= 2\Re \left( \tilde{\mathbf{E}}(\hat{q}), \tilde{\mathbf{E}}(\tilde{q}) \right)_{L^2(\Omega^{obs})} \\ &= 2\Re \left\langle \tilde{\mathbf{E}}|_{\Omega^{obs}}^* \tilde{\mathbf{E}}(\hat{q})|_{\Omega^{obs}}, \tilde{q} \right\rangle_{(C_0^{m,\alpha}(\Omega))' \times C_0^{m,\alpha}(\Omega)}, \end{aligned}$$

where  $(\cdot)^*$  denotes the adjoint operator. Note that the composition  $\tilde{\mathbf{E}}|_{\Omega^{obs}}^* \tilde{\mathbf{E}}|_{\Omega^{obs}}$  is meaningful since  $C^{m-1,\alpha}(\Omega^{obs}) \subset (C^{m-1,\alpha}(\Omega^{obs}))'$ ; in fact it is continuous. In this form, the first part of the Gauss-Newton Hessian

$$\mathcal{H}_1(q)(\cdot, \cdot) := \mathcal{H}_1(q; \cdot, \cdot) = 2\Re \left\langle \tilde{\mathbf{E}}|_{\Omega^{obs}}^* \tilde{\mathbf{E}}(\cdot)|_{\Omega^{obs}}, \cdot \right\rangle_{(C_0^{m,\alpha}(\Omega))' \times C_0^{m,\alpha}(\Omega)}$$

is evidently compact due to the compactness of  $\tilde{\mathbf{E}}(\cdot)|_{\Omega^{obs}}$ . To study the second part of the Gauss-Newton Hessian, we first write  $\tilde{\mathbf{H}}(\tilde{q})$  as

$$\begin{aligned} \tilde{\mathbf{H}}(\tilde{q}) &= \frac{1}{ik} \nabla \times \tilde{\mathbf{E}}(\tilde{q}) \\ &= ik \nabla \times \int_{\Omega} q(\mathbf{y}) \tilde{\mathbf{E}}(\tilde{q}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} + ik \nabla \times \int_{\Omega} \tilde{q}(\mathbf{y}) (\mathbf{E} + \mathbf{E}^{ic}) \Phi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \end{aligned}$$

where we have used the fact that  $\nabla \times \nabla(\cdot) = 0$ . Since  $\tilde{\mathbf{E}}(\tilde{q})$  maps  $C_0^{m,\alpha}(\Omega)$  continuously into  $C^{m,\alpha}(\Omega)$ , we conclude that  $\tilde{\mathbf{H}}(\tilde{q})$ , as a operator, is continuous from  $C_0^{m,\alpha}(\Omega)$  to  $C^{m+1,\alpha}(\Omega)$  owing to  $\mathbf{E} \in C^{m,\alpha}(\Omega)$ ,  $q, \tilde{q} \in C_0^{m,\alpha}(\Omega)$ , and Lemma 4.1. By the standard Hölder compact embeddings [21],  $\tilde{\mathbf{H}}(\tilde{q})$  is a compact operator from  $C_0^{m,\alpha}(\Omega)$  to  $C^{m,\alpha}(\Omega)$ . Argue similarly as above, we deduce that the second part of the Gauss-Newton Hessian

$$\mathcal{H}_2(q)(\cdot, \cdot) := \mathcal{H}_2(q; \cdot, \cdot) = 2\Re \left\langle \tilde{\mathbf{H}}|_{\Omega^{obs}}^* \tilde{\mathbf{H}}(\cdot)|_{\Omega^{obs}}, \cdot \right\rangle_{(C_0^{m,\alpha}(\Omega))' \times C_0^{m,\alpha}(\Omega)}$$

is a compact operator.

Let us now study the Gauss-Newton Hessian with pointwise observations. On the one hand, we rewrite the right side of (19) as

$$\begin{aligned}
 & -\nabla_{\mathbf{x}} \int_{\Omega} \nabla_{\mathbf{y}} \cdot (\tilde{q}(\mathbf{E} + \mathbf{E}^{ic})) \frac{\Phi(\mathbf{x}, \mathbf{y})}{n(\mathbf{y})} d\mathbf{y} - k^2 \int_{\Omega} \tilde{q}(\mathbf{E} + \mathbf{E}^{ic}) \Phi(\mathbf{x}, \mathbf{y}) d\mathbf{y} \\
 & = \int_{\Omega} \underbrace{\left[ (\mathbf{E} + \mathbf{E}^{ic}) \cdot \nabla_{\mathbf{y}} \left( \frac{\Phi'_{\mathbf{x}}(\mathbf{x}, \mathbf{y})}{n(\mathbf{y})} \right) - k^2 (\mathbf{E} + \mathbf{E}^{ic}) \Phi(\mathbf{x}, \mathbf{y}) \right]}_{\mathbf{f}_1(\mathbf{x}, \mathbf{y})} \tilde{q}(\mathbf{y}) d\mathbf{y},
 \end{aligned}$$

where integrating the first term by parts, using compact support of  $\tilde{q}$ , and interchanging the order of integration and differentiation have been used. Here,  $\Phi'_{\mathbf{x}} = \nabla_{\mathbf{x}}\Phi$  and  $\nabla_{\mathbf{y}}$  is understood to act component-wise.

On the other hand, using the adjoint  $\tilde{\mathbf{E}}^*$  allows us to write

$$\mathcal{T}[q]\tilde{\mathbf{E}}(\mathbf{x}) = - \left\langle \underbrace{\tilde{\mathbf{E}}^* \left( \frac{\Phi'_{\mathbf{x}}(\mathbf{x}, \mathbf{y})}{1 - q(\mathbf{y})} \otimes \nabla q(\mathbf{y}) \right)}_{\mathbf{f}_2(\mathbf{x}, \mathbf{y})}, \tilde{q}(\mathbf{y}) \right\rangle - \left\langle \underbrace{\tilde{\mathbf{E}}^* (k^2 q(\mathbf{y}) \Phi(\mathbf{x}, \mathbf{y}))}_{\mathbf{f}_3(\mathbf{x}, \mathbf{y})}, \tilde{q}(\mathbf{y}) \right\rangle,$$

with  $\langle \cdot, \cdot \rangle$  denoting the duality pairing between  $(C_0^{m,\alpha}(\Omega))'$  and  $C_0^{m,\alpha}(\Omega)$ . Consequently, from incremental forward integral equation (19), the evaluation of  $\tilde{\mathbf{E}}(\tilde{q})$  at  $\mathbf{x}_j^{obs}$  can be written as

$$\tilde{\mathbf{E}}(\tilde{q})(\mathbf{x}_j^{obs}) = \left\langle \underbrace{[\mathbf{f}_1(\mathbf{x}_j^{obs}, \mathbf{y}) - \mathbf{f}_2(\mathbf{x}_j^{obs}, \mathbf{y}) - \mathbf{f}_3(\mathbf{x}_j^{obs}, \mathbf{y})]}_{\Psi_j(\mathbf{y})}, \tilde{q}(\mathbf{y}) \right\rangle.$$

It follows that the first part of the Gauss-Newton Hessian can be written as

$$\mathcal{H}_1(q)(\hat{q}, \tilde{q}) = 2\Re \sum_{j=1}^{N^{obs}} \tilde{\mathbf{E}}(\tilde{q})(\mathbf{x}_j^{obs}) \overline{\tilde{\mathbf{E}}(\hat{q})(\mathbf{x}_j^{obs})} = 2\Re \left\langle \left\langle \sum_j^{N^{obs}} \overline{\Psi_j} \Psi_j, \tilde{q} \right\rangle, \hat{q} \right\rangle,$$

which shows that the dimension of the range of  $\mathcal{H}_1(q)$  is at most  $N^{obs}$ . As a result,  $\mathcal{H}_1(q)$  is a compact operator. By the same token, one can show that the second part of the Gauss-Newton Hessian  $\mathcal{H}_2(q)(\hat{q}, \tilde{q})$  is compact due to its finite dimensional range.

We summarize the above result on the compactness of the Gauss-Newton Hessian in the following theorem which is valid for both continuous and pointwise observation cases.

**Theorem 5.2.** *The Gauss-Newton Hessian,  $\mathcal{H}_1(q)$  plus  $\mathcal{H}_2(q)$ , as a continuous bilinear form on  $C_0^{m,\alpha}(\Omega) \times C_0^{m,\alpha}(\Omega)$ , is compact.*

The analysis of  $\mathcal{H}_3(q)$  is somewhat easier as we shall now show.

**Theorem 5.3.**  *$\mathcal{H}_3(q)$ , as a continuous bilinear form on  $C_0^{m,\alpha}(\Omega) \times C_0^{m,\alpha}(\Omega)$ , is a compact operator.*

*Proof.* Rewrite  $\mathcal{H}_3(q; \hat{q}, \tilde{q})$  as

$$\mathcal{H}_3(q; \hat{q}, \tilde{q}) = 2k\Re \int_{\Omega} \left[ \overline{i\mathbf{e}} \cdot \tilde{\mathbf{E}}(\tilde{q})\hat{q} + \overline{i\mathbf{e}} \cdot \tilde{\mathbf{E}}(\hat{q})\tilde{q} \right] d\Omega = 2k\Re \left\langle \overline{i\mathbf{e}} \cdot \overline{\tilde{\mathbf{E}}(\tilde{q})} + \tilde{\mathbf{E}}^*(\overline{i\mathbf{e}\tilde{q}}), \hat{q} \right\rangle$$

We conclude that  $\mathcal{H}_3(q)$  is compact by the following three observations. First, the incremental forward solution  $\tilde{\mathbf{E}}$  can be identified as a compact operator as discussed above. Second, multiplication by  $\overline{i\mathbf{e}}$  is a continuous operation in  $C^{m,\alpha}(\Omega)$  (see [5] for example). Third, the sum of two compact operators is again compact.  $\square$

We close this section by observing that the full Hessian is the sum of three compact operators, it is therefore compact as well.

**6. Analysis of the Hessian in Sobolev spaces.** Similar to our previous work [5, 6], we shall extend the analysis in Hölder spaces to Sobolev spaces. We recall a result [6] on the mapping property of  $T$  in Sobolev spaces that is result similar to that of Lemma 4.1.

**Lemma 6.1.** *Assume that  $\varphi$  is bounded and integrable,  $\Omega \subset \mathbb{R}^3$  is a bounded domain. Then  $T$  defined in (15) maps  $H^m(\Omega)$  continuously to  $H^{m+2}(\Omega)$  for  $m \in \mathbb{N} \cup \{0\}$ .*

We next summarize main results parallel to those in Sections 4 and 5. Since the proofs are similar, we omit the details.

**Corollary 2.** *Suppose  $q, \tilde{q} \in H^m(\Omega)$ ,  $m \in \mathbb{N}$ .*

- i)  $\mathcal{Q}[\tilde{q}]$  maps  $H^m(\Omega)$  continuously into  $H^m(\Omega)$ .*
- ii)  $\mathcal{T}[q], \hat{\mathcal{T}}[\tilde{q}]$  map  $H^{m-1}(\Omega)$  continuously into  $H^m(\Omega)$ , and  $\mathcal{T}[q], \hat{\mathcal{T}}[\tilde{q}] : H^{m-1}(\Omega) \rightarrow H^p(\Omega)$  are compact operators for  $m - 1 \leq p < m$ .*

Note that the last result in Corollary 2 is due to compact embeddings in Sobolev spaces [2, 18]. From now on to the end of this section, we need  $m \geq 2$  for point observations to make sense since only Sobolev spaces of order greater than 3/2 are embedded into the space of continuous functions [18].

**Theorem 6.2.** *Let  $q, \tilde{q} \in H^m(\Omega)$ ,  $m \in \mathbb{N}$  and  $m \geq 2$ , there exists a unique solution to*

- i) the forward integral equation (18) such that  $\mathbf{E} \in H^m(\Omega)$ ,  $\mathbf{H} \in H^{m+1}(\Omega)$ , and*

$$\begin{aligned} \|\mathbf{E}\|_{H^m(\Omega)} &\leq c \|q\|_{H^m(\Omega)} \|\mathbf{E}^{ic}\|_{H^m(\Omega)}, \\ \|\mathbf{H}\|_{H^{m+1}(\Omega)} &\leq c \|q\|_{H^m(\Omega)} \|\mathbf{E} + \mathbf{E}^{ic}\|_{H^m(\Omega)}. \end{aligned}$$

- ii) the incremental forward integral equation (19) such that  $\tilde{\mathbf{E}} \in H^m(\Omega)$ ,  $\tilde{\mathbf{H}} \in H^{m+1}(\Omega)$ , and  $\|\tilde{\mathbf{H}}\|_{H^{m+1}(\Omega)} \leq c \|\tilde{\mathbf{E}}\|_{H^m(\Omega)}$  where*

$$\|\tilde{\mathbf{E}}\|_{H^m(\Omega)} \leq c \|\tilde{q}\|_{H^m(\Omega)} \left( \|\mathbf{E}\|_{H^m(\Omega)} + \|\mathbf{E}^{ic}\|_{H^m(\Omega)} \right).$$

- iii) the adjoint integral equation (20) such that  $\mathbf{e} \in H^m(\Omega)$  and  $\|\mathbf{e}\|_{H^m(\Omega)} \leq c (\|\mathbf{E} - \mathbf{E}^{obs}\|_{\infty} + \|\mathbf{H} - \mathbf{H}^{obs}\|_{\infty})$ .*

- iv) the incremental adjoint integral equation (21) such that  $\tilde{\mathbf{e}} \in H^m(\Omega)$  and  $\|\tilde{\mathbf{e}}\|_{H^m(\Omega)} \leq c \left( \|\tilde{\mathbf{E}}\|_{\infty} + \|\tilde{\mathbf{H}}\|_{\infty} + \|\mathbf{e}\|_{H^m(\Omega)} \right)$ .*

Clearly, the Sobolev setting also shows that the gradient and the Hessian are meaningful.

**Theorem 6.3.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain. Assume that  $q, \hat{q}$  and  $\tilde{q}$  belong to  $H^m(\Omega)$  with  $m \in \mathbb{N}$  and  $m \geq 2$ . Then, the cost functional (2) is twice continuously Fréchet differentiable; and hence the gradient (6) and Hessian (14) are well-defined.*

**Lemma 6.4.** *Let  $q, \tilde{q} \in H^m(\Omega)$ ,  $m \in \mathbb{N}$  and  $m \geq 2$ . Then  $\tilde{\mathbf{E}} : H^m(\Omega) \rightarrow H^{m-1}(\Omega)$  is a compact operator.*

Here comes the main result of this section.

**Theorem 6.5.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain. Suppose  $q \in H^m(\Omega)$ ,  $m \in \mathbb{N}$  and  $m \geq 2$ . Then, the Hessian,  $\mathcal{H}(q) = \mathcal{H}_1(q) + \mathcal{H}_2(q) + \mathcal{H}_3(q)$ , is a compact operator in  $H^m(\Omega)$ .*

**7. Conclusions.** We have analyzed the Hessian stemming from the inverse problem of scattering of electromagnetic waves due to bounded inhomogeneity. Similar to our companion paper on inverse medium scattering problems of acoustic waves [6], we have shown that the full Hessian is a compact operator. Our analysis starts with a study on the regularity of the scattering solution with respect to the smoothness of the medium based on the Newton potential theory and the Riesz-Fredholm framework. Then, together with compact embeddings in Hölder and Sobolev spaces, we are able to prove the compactness of the Hessian operator in both Hölder space and Sobolev space settings for three dimensional inverse medium electromagnetic scattering problems. Extending our results to the case of less smooth refractive index  $n$  (and hence  $q$ ) is interesting, though more technical, and it is a subject for our future work.

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