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A UNIFIED DISCONTINUOUS PETROV–GALERKIN METHOD AND ITS ANALYSIS FOR FRIEDRICHS' SYSTEMS*

TAN BUI-THANH†, LESZEK DEMKOWICZ†, AND OMAR GHATTAS‡

Abstract. We propose a unified discontinuous Petrov–Galerkin (DPG) framework with optimal test functions for Friedrichs-like systems, which embrace a large class of elliptic, parabolic, and hyperbolic partial differential equations (PDEs). The well-posedness, i.e., existence, uniqueness, and stability, of the DPG solution is established on a single abstract DPG formulation, and two abstract DPG methods corresponding to two different, but equivalent, norms are devised. We then apply the single DPG framework to several linear(ized) PDEs including, but not limited to, scalar transport, Laplace, diffusion, convection-diffusion, convection-diffusion-reaction, linear(ized) continuum mechanics (e.g., linear(ized) elasticity, a version of linearized Navier–Stokes equations, etc.), time-domain acoustics, and a version of the Maxwell's equations. The results show that we not only recover several existing DPG methods, but also discover new DPG methods for both PDEs currently considered in the DPG community and new ones. As a direct consequence of the single abstract DPG framework, all of the resulting DPG methods are shown to be trivially well-posed. We show that the inf-sup constant of the abstract DPG equation is independent of the mesh and is the same order as that of the PDE counterpart.

Key words. discontinuous Petrov–Galerkin methods, well-posedness, partial differential equations, Friedrichs' systems, inf–sup condition, consistency, stability, convergence

AMS subject classifications. 65N30, 65N12, 65N15, 65N22

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1. Introduction.

The discontinuous Petrov–Galerkin (DPG) framework introduced by Demkowicz and Gopalakrishnan [13, 15] has been evolving as a new numerical methodology for partial differential equations (PDEs). The method has been successfully applied to a wide range of PDEs including scalar transport [8, 13, 15], Laplace [14], convection-diffusion [14, 15], Helmholtz [16, 18, 30], Burgers and Navier–Stokes [10], and linear elasticity [7] equations. This DPG framework starts by partitioning the domain of interest into nonoverlapping elements. Variational formulations are posed for each element separately and then summed up to form a global variational statement. Elemental solutions are connected by introducing hybrid variables (also known as fluxes or traces) that live on the skeleton of the mesh. This is therefore a mesh-dependent variational approach in which both bilinear and linear forms depend on the mesh under consideration.

In general, the trial and test spaces are not related to each other. In the standard Bubnov–Galerkin (also known as Galerkin) approach, the trial and test spaces are identical, while they differ in a Petrov–Galerkin scheme. Traditionally, one chooses either Galerkin or Petrov–Galerkin approaches, then proves the consistency and stability in both infinite and finite dimensional settings (if possible). The DPG method of Demkowicz and Gopalakrishnan [13, 15] introduces a new paradigm in which one selects both trial and test spaces at the same time to satisfy well-posedness. In partic-

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†Institute for Computational Engineering & Sciences, The University of Texas at Austin, Austin, TX 78712 (tanbui@ices.utexas.edu, leszek@ices.utexas.edu).

‡Institute for Computational Engineering & Sciences, Jackson School of Geosciences, and Department of Mechanical Engineering, The University of Texas at Austin, Austin, TX 78712 (omar@ices.utexas.edu).

ular, one can select trial and test function spaces for which the continuity and inf–sup constants are unity. Given a finite dimensional trial subspace, the finite dimensional test space is constructed in such a way that the well-posedness of the finite dimensional setting is automatically inherited from the infinite dimensional counterpart.

For example, the DPG method in [15] starts with a given norm in the trial space and then seeks a norm in the test space in order to achieve unity continuity and inf–sup constants. Another DPG method in [16] achieves the same goal but reverses the process, i.e., it looks for a norm in the trial space corresponding to a given norm in the test space. Clearly, this is one of the advantages of the DPG methodology, since it allows one to choose a norm of interest to work with, while rendering the error optimal, i.e., smallest in that norm. Furthermore, the DPG methodology provides a natural framework for constructing robust versions of the method for singular perturbation problems, enabling automatic adaptivity. We shall not discuss the advantages of the DPG methods any further here, and the readers are referred to the original DPG papers [13, 14, 15, 16] for more details.

The DPG method is a minimum residual method and can be viewed as a generalization of least squares methods [5, 9]. The main difference lies in the use of dual norms through an explicit elementwise inversion of the Riesz operator made possible by the use of broken test spaces as opposed to Hilbert scalings in [6].

Perhaps one of the most challenging problems that needs to be addressed in developing a DPG method is to establish the well-posedness of the DPG formulation on the infinite dimensional level, from which the well-posedness of a finite dimensional DPG approximation is inherited. This has been investigated for DPG formulations of linear first order hyperbolic [8], Laplace [14], convection-diffusion [14], Helmholtz [18], and linear elasticity [7] equations. The methods of proof however vary from one type of PDE to another, though they do share some similarities. Consequently, practitioners may be wary of applying the DPG methodology to a new PDE until its well-posedness is available. Otherwise, there is no guarantee that a DPG method would behave as designed in the original work of Demkowicz and Gopalakrishnan [13, 15].

Meanwhile, a unified analysis of discontinuous Galerkin (DG) methods for elliptic/parabolic/hyperbolic PDEs and beyond has been devised in a series of papers by Ern and Guermond [21, 22, 23]. This is possible due to the recent revised theory of Friedrichs' system [25] in a Hilbert space setting [27], rigorously formalized and advanced by [24], and further advanced by [1, 2, 3]. Ern and Guermond [21, 22, 23] have been successful in recovering most of the existing DG methods and discovering new ones for various PDEs including transport, convection-diffusion-reaction, linear(ized) continuum mechanics, and Maxwell's equations, to name a few.

The success of Ern and Guermond [21, 22, 23] inspires and motivates us to develop a unified theory for the DPG methodology for a large class of PDEs, and this is the main focus of the paper. In particular, we review the theory of Friedrichs-like systems under a Hilbert space setting [24] in section 2.1. In particular, section 2.2 reviews Friedrichs' systems of first order PDEs, followed by Friedrichs' systems of first order PDEs with partial coercivity in section 2.2.2 with the important result on the well-posedness in Theorem 2.3. We next develop a single abstract DPG framework, prove its well-posedness, and derive two abstract DPG methods corresponding to two different, but equivalent, norms in section 2.3. It is followed by the convergence analysis of DPG methods in section 2.4. Note that this paper is neither an attempt to illuminate connections with other related ideas and methods nor an attempt to unify all the existing DPG methods. Instead, we limit ourself to unify and generalize the DPG methods of Demkowicz and Gopalakrishnan [13, 15] for Friedrichs' PDE systems. To show the effectiveness of the single abstract framework, section 3 applies it to various PDEs including transport, convection-diffusion-reaction, linear(ized) continuum mechanics, time-domain acoustic, and a version of the Maxwell's equations. As will be shown, our unified framework not only recovers several existing DPG methods, but also discovers new DPG methods for both PDEs currently considered in the DPG community and new ones. More importantly, a single well-posedness proof established for the abstract and unified DPG methodology is carried over to all Friedrichs-like systems in general and to all PDEs considered in section 3 in particular. Finally, section 4 concludes the paper with future directions.

2. Abstract theory.

2.1. Theory of Friedrichs' systems in a Hilbert space setting. In this section, we briefly review important theoretical advances of Friedrichs' systems in Hilbert space settings due to Ern, Guermond, and Caplain [24] that are useful for our later developments. To begin, let L be a real Hilbert space equipped with the inner product $(\cdot,\cdot)_L$ and the induced norm $\|\cdot\|_L$. We identify L with its dual L' by the Riesz representation theorem. Assume that we have two linear operators (possibly unbounded) $T : \mathscr{D} \to L$ and $\tilde{T} : \mathscr{D} \to L$ satisfying the following two properties:

$$(T\varphi,\psi)_L = \left(\varphi,\tilde{T}\psi\right)_L \qquad \forall\varphi,\psi\in\mathscr{D}, \tag{2.1a}$$

$$\left\|\left(T+\tilde{T}\right)\varphi\right\|_L \le c\,\|\varphi\|_L \qquad \forall\varphi\in\mathscr{D}, \tag{2.1b}$$

where $\mathscr{D}$ is a dense subspace of L. Note that, by density, (2.1b) is also valid for all $\varphi \in L$.

It is easy to see that $\mathscr{D}$ equipped with the scalar product $(\cdot,\cdot)_T = (\cdot,\cdot)_L + (T\cdot,T\cdot)_L$ is an inner product space whose completion is denoted by W_0. The induced norm $\|\cdot\|_T = \sqrt{(\cdot,\cdot)_L + (T\cdot,T\cdot)_L}$ is known as the graph norm. One can show that the completion of $\mathscr{D}$ with respect to $(\cdot,\cdot)_{\tilde{T}} = (\cdot,\cdot)_L + (\tilde{T}\cdot,\tilde{T}\cdot)_L$ coincides with W_0. As a direct consequence, $T,\tilde{T} : (\mathscr{D},\|\cdot\|_T) \to (L,\|\cdot\|_L)$ are linear and continuous, and hence they can be extended by density to linear and continuous operators (again denoted by T and $\tilde{T}$) $T,\tilde{T} : (W_0,\|\cdot\|_T) \to (L,\|\cdot\|_L)$. Also by density, (2.1) can be extended to be valid for all $\varphi,\psi \in W_0$. Moreover, it can be shown that the adjoints of T and $\tilde{T}$ are the unique extensions of $\tilde{T}$ and T, again denoted by $\tilde{T}$ and T such that $\tilde{T}, T : L \to W_0'$ and

$$\langle Tu, v\rangle_{W_0'\times W_0} = \left(u,\tilde{T}v\right)_L \quad \forall u\in L, v\in W_0, \tag{2.2a}$$

$$\left\langle \tilde{T}u, v\right\rangle_{W_0'\times W_0} = (u,Tv)_L \quad \forall u\in L, v\in W_0, \tag{2.2b}$$

where $\tilde{T}$ and T on the right sides of (2.2a) and (2.2b) should be understood as the restrictions of $\tilde{T}$ and T, i.e., $\tilde{T}|_{W_0}, T|_{W_0} : W_0 \to L$.

We are interested in the solvability of the problem

$$Tu = f \in L, \tag{2.3}$$

and its solutions generally belong to the following graph space

$$W := \{u \in L : Tu \in L\},$$

which can be shown to coincide with the dual graph space $\{v \in L : \tilde{T}v \in L\}$. It is not difficult to see that W is a Hilbert space when equipped with the graph inner product $(\cdot,\cdot)_W := (\cdot,\cdot)_T$. However, the graph space is too general to provide the well-posedness of (2.3) and our next step is to find a subspace $V \subseteq W$ such that $T : V \to L$ is an isomorphism. We begin by defining the following *boundary operator*:

$$\langle Bu, v\rangle_{W'\times W} := (Tu, v)_L - \left(u, \tilde{T}v\right)_L \quad \forall u, v \in W. \tag{2.4}$$

Then, one can show that $B \in \mathscr{L}(W, W')$ and B is self-adjoint [24].

Now, assume there exists $M \in \mathscr{L}(W, W')$ such that

$$\langle Mw, w\rangle \geq 0 \quad \forall w \in W, \tag{2.5a}$$

$$W = \mathcal{N}(B - M) + \mathcal{N}(B + M) \tag{2.5b}$$

with $\mathcal{N}$ denoting the nullspace of its argument. As shall be shown in section 3, each M corresponds to a particular boundary condition and it may not be unique. The following useful result on B and M is due to [24].

THEOREM 2.1. *There hold*

$$W_0 = \mathcal{N}(B) = \mathcal{N}(M) = \mathcal{N}(M^*), \tag{2.6a}$$

$$W_0^\perp = \mathcal{R}(B) = \mathcal{R}(M) = \mathcal{R}(M^*), \tag{2.6b}$$

where $\mathcal{R}$ denotes the range space.

2.2. First order partial differential equations of Friedrichs' type. The results in section 2 are valid for a general class of operators T and $\tilde{T}$ satisfying (2.1a), (2.1b) on a dense subset $\mathscr{D}$ of Hilbert space L. In this section and in the rest of the paper, we restrict ourselves to L as the space of square integral (vector-valued) functions over an open and bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, $\mathscr{D}$ as the space of test functions, and T as first order differential operator with its formal adjoint $\tilde{T}$. In particular, let us set $L = \left[L^2(\Omega)\right]^m$, $m \in \mathbb{N}$, and $\mathscr{D} = [\mathcal{D}(\Omega)]^m$, where $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$. Then, $\mathscr{D}$ is dense in L.

Next, we consider $T : \mathscr{D} \to L$ as

$$T\varphi := \sum_{k=1}^d A^k \partial_k \varphi + C\varphi \quad \forall \varphi \in \mathscr{D},$$

where the following assumptions on A^k and C for Friedrichs' system are standard [21, 25]:

$$C \in [L^\infty(\Omega)]^{m,m}, \tag{2.7a}$$

$$A^k \in [L^\infty(\Omega)]^{m,m}, \quad k = 1, \ldots, d, \quad \text{and} \quad \sum_{k=1}^d \partial_k A^k \in [L^\infty(\Omega)]^{m,m}, \tag{2.7b}$$

$$A^k = \left(A^k\right)^T \text{ a.e. in } \Omega, \quad k = 1, \ldots, d. \tag{2.7c}$$

Consequently, the formal adjoint $\tilde{T} : \mathscr{D} \to L$ of T is given by

$$\tilde{T}\varphi = -\sum_{k=1}^d A^k \partial_k \varphi + \left(C^* - \sum_{k=1}^d \partial_k A^k\right)\varphi \quad \forall \varphi \in \mathscr{D}.$$

Then, it is obvious to see that T and $\tilde{T}$ satisfy (2.1a), (2.1b). Consequently, all the results in section 2 hold for Friedrichs' systems satisfying (2.7).

For the above abstract Friedrichs' systems, an explicit representation of B is available while it is not possible for M on the abstract level. Let us assume that

$$\mathcal{B} := \sum_{k=1}^{d} n_k A^k$$

is well-defined a.e. on $\partial\Omega$ with $\mathbf{n} = (n_1, \ldots, n_d)$ being the unit outward normal vector of $\partial\Omega$. For simplicity in writing, let us set $\mathcal{H}^s := [H^s]^m$, where H^s is the usual Sobolev space of order s, and $\mathcal{C}^1 := \left[C^1\right]^m$, where C^1 is the space of continuously differentiable functions. The following representation result for the boundary operator B can be found in [1, 27].

LEMMA 2.2. *For $u, v \in \mathcal{H}^1(\Omega) \subset W(\Omega)$, there holds*

$$\langle Bu, v\rangle_{W'(\Omega)\times W(\Omega)} = \langle \mathcal{B}u, v\rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)\times\mathcal{H}^{\frac{1}{2}}(\partial\Omega)}.$$

In particular, for $u, v \in \mathcal{C}_0^\infty\left(\mathbb{R}^d\right)$, $\langle Bu, v\rangle_{W'(\Omega)\times W(\Omega)} = \int_{\partial\Omega} v^T \mathcal{B}u \, ds$.

If Ω has segment property [1], which is true for Lipschitz domains, then $\mathcal{C}^1$ is dense in $\mathcal{H}^1(\Omega)$ which in turn is dense in W, and hence the representation can be uniquely extended to the whole space W, i.e.,

$$\langle Bu, v\rangle_{W'(\Omega)\times W(\Omega)} = \langle \mathcal{B}u, v\rangle_{\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)\times\mathcal{H}^{\frac{1}{2}}(\partial\Omega)} \forall u \in W(\Omega), v \in \mathcal{H}^1(\Omega). \tag{2.8}$$

Definition (2.4) can be therefore considered as the integration by parts formula. It is important to point out that the map $\mathcal{B} : W(\Omega) \to \mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$ is not surjective in general [1]. Moreover, the range of $\mathcal{B}$ is generally not closed in $\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$. Owing to this fact, the boundary operator B may be more preferable since its nullspace W_0 is well defined and its range space $W_0^\perp$ is obviously closed. It is the key that we explore in this paper. In particular, the construction of the abstract DPG method (2.14) using the boundary operator is twofold. First, we avoid the technicality of specifying the trace of functions in an abstract graph space $W(\Omega)$,[1] allowing the DPG theory to be developed for abstract operators T and $\tilde{T}$. Second, the well-posedness of the resulting general DPG method can be established in a straightforward manner.

On the one hand, Lemma 2.2 supports our abstract DPG formulation (2.14) in using boundary operator and graph space, which is valid for a general Friedrichs' differential operator T even when we do not have a trace theorem for the graph space. On the other hand, as shall be shown in section 3, the representation result in Lemma 2.2 allows us to solve for the unknown hybrid variables on the skeleton of the mesh instead of the whole domain, which results in substantial savings in computation.

In order the show the well-posedness of PDEs of Friedrichs' type we need the coercivity condition on T dictated by the positiveness condition on the coefficients A_k and C [21, 22, 23]. To this end, we consider two classes of first order PDEs: one with full coercivity and the other with partial coercivity.

2.2.1. Friedrichs' PDEs with full coercivity. By full coercivity we mean the following positiveness condition:

$$C + C^* - \sum_{k=1}^{d} \partial_k A^k \gtrsim I_m \text{ a.e. in } \Omega, \tag{2.9}$$

[1]The technicality here is due the fact that v belongs to the broken graph space and its trace also lives in $\mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$. Consequently, the duality between the traces of u and v is not meaningful in the standard $\mathcal{H}^{-\frac{1}{2}}(\partial\Omega) \times \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$ sense.

where I_m is the $m \times m$ identity matrix. Here, the notation $z \gtrsim u$ (similarly for $z \lesssim u$) means $z \geq \alpha u$ for some positive constant α.

2.2.2. Friedrichs' PDEs with partial coercivity. For the class with partial coercivity, we relax the positivity condition (2.9) to account for systems that have two-field structures with partial coercivity. This class includes convection-diffusion, Laplace, and linearized continuum mechanics (e.g., linearized compressible elasticity or linearized compressible Navier–Stokes) equations, to name a few. Following [23], we assume that there exist two positive integers $m_{\boldsymbol{\sigma}}$ and m_u such that $m = m_{\boldsymbol{\sigma}} + m_u$. Denote $L_{\boldsymbol{\sigma}} := \left[L^2(\Omega)\right]^{m_{\boldsymbol{\sigma}}}$, $L_u := \left[L^2(\Omega)\right]^{m_u}$, and $L := L_{\boldsymbol{\sigma}} \times L_u$. For any $w \in L$, the group variable notion $w := (w^{\boldsymbol{\sigma}}, w^u)$ is used throughout. We decompose C and A accordingly:

$$C = \begin{bmatrix} C^{\boldsymbol{\sigma}\boldsymbol{\sigma}} & C^{\boldsymbol{\sigma}u} \\ C^{u\boldsymbol{\sigma}} & C^{uu} \end{bmatrix}, \quad A^k = \begin{bmatrix} A^{\boldsymbol{\sigma}\boldsymbol{\sigma},k} & E^k \\ \left(E^k\right)^T & G^k \end{bmatrix}.$$

The following assumptions are important for the well-posedness of our two-field Friedrichs' systems with partial coercivity [23]:

$$\forall k \in 1, \ldots, d, \quad A^{\boldsymbol{\sigma}\boldsymbol{\sigma},k} = 0, \tag{2.10a}$$

$$\exists c_0 > 0, \quad C^{\boldsymbol{\sigma}\boldsymbol{\sigma}} \geq c_0 I_{m_{\boldsymbol{\sigma}}}, \tag{2.10b}$$

$$\left(\left(C + C^* - \sum_{k=1}^{d} \partial_k A^k\right) z, z\right) \gtrsim \|z^{\boldsymbol{\sigma}}\|_{L_{\boldsymbol{\sigma}}}^2 \text{ a.e. in } \Omega, \tag{2.10c}$$

$$C^{\boldsymbol{\sigma}u} = \left(C^{u\boldsymbol{\sigma}}\right)^* = 0 \text{ and } E^k \text{ are constant over } \Omega, \tag{2.10d}$$

$$\forall z \in V \cup V^*, \|z^u\|_{L_u} \lesssim \tilde{b}(z, z)^{\frac{1}{2}} + \|E z^u\|_{L_{\boldsymbol{\sigma}}}, \tag{2.10e}$$

where

$$\tilde{b}(u, v) := (Tu, v)_L + \frac{1}{2} \langle (M - B) u, v \rangle_{W' \times W}$$

and

$$E := \sum_{k=1}^{d} E^k \partial_k, \quad V := \mathcal{N}(B - M), \quad V^* := \mathcal{N}(B + M^*).$$

Note that the condition (2.10e) is meaningful owing to the positive definiteness of $\tilde{b}$ on W.

We are now in position to state the well-posedness whose proof can be found in [21, 22, 23, 24].

THEOREM 2.3. *Assume that either the full coercivity condition* (2.9) *or the partial coercivity condition* (2.10) *holds, then $T : V \to L$ is bijective. Furthermore, given $f \in L$, then the problem of seeking $u \in V$ such that $Tu = f$ in L is well-posed. In particular, there exists a positive constant μ_0 such that if u is the solution, then it satisfies the following stability estimates:*

$$\|u\|_L \leq \frac{1}{\mu_0} \|f\|_L, \quad \|u\|_W \leq \left(1 + \frac{1}{\mu_0}\right) \|f\|_L.$$

We have reviewed the Friedrichs' setting for PDEs with either full or partial coercivity. Theorem 2.3 on the well-posedness in both cases is vital since the well-posedness of our unified DPG formulation in section 2.3 relies heavily on this fact.

2.3. Abstract DPG formulation. We are interested in the following inhomogeneous problem:

$$\text{(2.11)} \qquad \begin{cases} \text{Given } g \in W, f \in L. \text{ Seek } u \in W \text{ such that} \\ Tu = f \text{ in } L, \text{ and} \\ (u-g) \in V = \mathcal{N}(B-M), \end{cases}$$

which is clearly well-posed by Theorem 2.3. An equivalent variational formulation of (2.11) can be written as

$$\text{(2.12)} \qquad \begin{cases} \text{Given } g \in W, f \in L. \text{ Seek } u \in W \text{ such that } \forall v \in W, \\ (Tu, v)_L + \frac{1}{2}\langle (M-B)\, u, v\rangle_{W' \times W} = (f, v)_L + \frac{1}{2}\langle (M-B)\, g, v\rangle_{W' \times W}. \end{cases}$$

Note that both formulations (2.11) and (2.12) are not popular, but appealing since they avoid taking the trace of functions in W, which may not be well defined in general. More important, they permit us to study the well-posedness of an abstract DPG method in a quite general setting. To begin, let us partition the domain Ω into N^{el} nonoverlapping elements $K_j, j = 1, \ldots, N^{\mathrm{el}}$ with Lipschitz boundaries such that $\Omega_h = \cup_{j=1}^{N^{\mathrm{el}}} K_j$ and $\overline{\Omega} = \overline{\Omega}_h$. Here, h is defined as $h = \max_{j \in \{1,\ldots,N^{\mathrm{el}}\}} diam\,(K_j)$. As a result, all the results (respectively, assumptions) in section 2 are valid elementwise. We will attach the domain under consideration to operators and spaces whenever it is necessary to avoid confusion. For example, B_{K_j} is the boundary operator defined in (2.4) when T and $\tilde{T}$ are restricted on K_j.

Decomposing the first term of the left side of (2.12) and using definition (2.4) elementwise, we obtain

$$\sum_{j=1}^{N^{\mathrm{el}}} \left(u, \tilde{T}v\right)_{L(K_j)} + \sum_{j=1}^{N^{\mathrm{el}}} \left\langle B_{K_j} u, v\right\rangle_{W'(K_j)\times W(K_j)} + \frac{1}{2}\langle (M-B)\, u, v\rangle_{W'(\Omega)\times W(\Omega)}$$
$$= \sum_{j=1}^{N^{\mathrm{el}}} (f, v)_{L(K_j)} + \frac{1}{2}\langle (M-B)\, g, v\rangle_{W'(\Omega)\times W(\Omega)},$$

where u appearing in the duality pairings in the second term of the left side is understood as the restriction of u on K_j.

Now, it is natural to seek u in $L(\Omega_h) = L(\Omega)$, but then definition (2.4) is no longer valid. Therefore, we define a new variable q living in the quotient space $\tilde{W}(\Omega) := W/Q(\Omega)$ with Q given by

$$Q := \{q \in W(\Omega) : a(q, v) = 0 \ \forall v \in W(\Omega_h)\},$$

where

$$a(q, v) := \sum_{j=1}^{N^{\mathrm{el}}} \left\langle B_{K_j} q, v\right\rangle_{W'(K_j)\times W(K_j)} + \frac{1}{2}\langle (M-B)\, q, v\rangle_{W'(\Omega_h)\times W(\Omega_h)}.$$

Here, $W(\Omega_h) := \{v : v|_{K_j} \in W(K_j)\}$ is the broken graph space with norm defined via $\|v\|^2_{W(\Omega_h)} := \sum_{j=1}^{N^{\mathrm{el}}} \|\tilde{T}v\|^2_{L(K_j)} + \|v\|^2_{L(\Omega_h)}$ and $W'(\Omega_h)$ is its topological dual. Clearly, Q is a closed subspace of $W(\Omega)$, and hence it is meaningful to define the standard quotient norm in $\tilde{W}(\Omega)$ as

$$\|q\|_{\tilde{W}} = \inf_{r \in W(\Omega): r-q \in Q} \|r\|_W \quad \forall q \in \tilde{W}(\Omega).$$

Before stating our abstract DPG formulation, we need to extend, using a version of the Hahn–Banach theorem [20] or any other valid continuous extensions, $(M-B)\,q$ and $(M-B)\,g$ from $W'(\Omega)$ to $W'(\Omega_h)$, again denoted by $(M-B)\,q$ and $(M-B)\,g$, respectively. Note that both extensions are, in general, not unique. We therefore impose the following compatibility condition:

$$\begin{array}{c}(M-B)\,q = (M-B)\,g \text{ in } W'(\Omega)\\ \Downarrow \\ (M-B)\,q = (M-B)\,g \text{ in } W'(\Omega_h)\,.\end{array} \tag{2.13}$$

At this level of abstraction, the use of the Hahn–Banach extension argument together with the compatibility condition is necessary for our theory to be rigorous. In practice, both conditions are often trivially satisfied as demonstrated in all examples in this paper.

Now, let us propose the following DPG formulation:

$$\begin{aligned}&\text{Given } g \in W(\Omega)\,, f \in W'(\Omega_h)\,. \text{ Seek } \ (u,q) \in L(\Omega_h) \times \tilde{W}(\Omega) \ \text{ such that}\\ &\sum_{j=1}^{N^{\mathrm{el}}} \left(u, \tilde{T}v\right)_{L(K_j)} + \sum_{j=1}^{N^{\mathrm{el}}} \left\langle B_{K_j} q, v\right\rangle_{W'(K_j)\times W(K_j)} + \frac{1}{2}\left\langle (M-B)\,q, v\right\rangle_{W'(\Omega_h)\times W(\Omega_h)}\\ &\quad = \sum_{j=1}^{N^{\mathrm{el}}} \langle f, v\rangle_{W'(K_j)\times W(K_j)} + \frac{1}{2}\left\langle (M-B)\,g, v\right\rangle_{W'(\Omega_h)\times W(\Omega_h)} \quad \forall v \in W(\Omega_h)\,,\end{aligned} \tag{2.14}$$

where we have relaxed the data f in the DPG formulation (2.14) to allow it to live in the dual space $W'(\Omega_h) \supset L(\Omega)$ of the broken graph space $W(\Omega_h)$.

For convenience, we shall equivalently write (2.14) in the usual form $b((u,q),v) = \ell(v)$, where the bilinear form $b((u,q),v)$ and the linear form $\ell(v)$ are obviously defined as the right and left sides of (2.14), respectively.

The first step is to study the consistency of our DPG formulation. That is, if the data are sufficiently smooth, the solution of (2.11) should be a solution of the DPG formulation and vice versa.

LEMMA 2.4 (consistency). *Assume* $f \in L(\Omega)$. *If* $u \in W(\Omega)$ *is a solution of* (2.11)*, then* $(u,u) \in L(\Omega_h) \times \tilde{W}(\Omega)$ *is a solution of the DPG equation* (2.14). *Conversely, if* $(u,q) \in L(\Omega_h) \times \tilde{W}(\Omega)$ *is a solution of* (2.14)*, then* u *is a solution of* (2.11).

Proof. Let u be the unique solution of (2.11) and set $q = u$. Using the compatibility condition (2.13) and (2.4) we conclude that $(u,q) = (u, q = u)$ solves the DPG formulation (2.14).

Conversely, taking $v \in W_0(\Omega)$ we have

$$\sum_{j=1}^{N^{\mathrm{el}}} \left\langle B_{K_j} q, v\right\rangle_{W'(K_j)\times W(K_j)} = \langle Bq, v\rangle_{W'(\Omega)\times W(\Omega)} = \langle q, Bv\rangle_{W'(\Omega)\times W(\Omega)} = 0,$$

where we have used (2.4), self-adjointness of B, and Theorem 2.1. Similarly, using the compatibility condition we observe

$$\begin{aligned}\left\langle (M-B)\,q, v\right\rangle_{W'(\Omega_h)\times W(\Omega_h)} &= \left\langle (M-B)\,q, v\right\rangle_{W'(\Omega)\times W(\Omega)}\\ &= \left\langle q, (M^*-B)\,v\right\rangle_{W'(\Omega)\times W(\Omega)} = 0,\end{aligned} \tag{2.15}$$

and by the same token $\langle (M-B)\, g, v\rangle_{W'(\Omega_h)\times W(\Omega_h)} = 0$. Consequently, (2.14) simplifies to

$$(f,v)_{L(\Omega)} = \left(u, \tilde{T}v\right)_{L(\Omega)} = \langle Tu, v\rangle_{W_0'(\Omega)\times W_0(\Omega)} \quad \forall v \in W_0(\Omega),$$

where we have used (2.2a) in the last equality. It follows that $Tu = f \in L(\Omega)$, i.e., $u \in W(\Omega)$. What remains to be done is to show that $(u-g) \in V = \mathcal{N}(B-M)$.

Using (2.4) and taking $v \in W(\Omega)$, formulation (2.14) becomes

$$\langle B(q-u), v\rangle_{W'(\Omega)\times W(\Omega)} = \frac{1}{2}\langle (M-B)(g-q), v\rangle_{W'(\Omega)\times W(\Omega)},$$

and hence

$$B(q-u) = (M-B)\frac{(g-q)}{2} \text{ in } W'(\Omega). \tag{2.16}$$

Now, given (2.5a), it can be shown, see [24, Lemma 4.2], that (2.5b) is equivalent to

$$W = \mathcal{N}(B-M^*) + \mathcal{N}(B+M^*),$$

which, after using a similar argument as in [24, Lemma 4.3], implies

$$\overline{\mathcal{R}(B-M)} \cap \overline{\mathcal{R}(B+M)} = \{0\}.$$

Since $\mathcal{R}(B) = \mathcal{R}(M)$ as stated in Theorem 2.1, it follows that

$$\overline{\mathcal{R}(B-M)} \cap \overline{\mathcal{R}(B)} = \{0\}. \tag{2.17}$$

Combining (2.16) and (2.17) yields

$$(B-M)(u-g) = 0,$$

and hence u is a solution of (2.11). □

COROLLARY 2.5. *Assume $f \in L(\Omega)$. There exists a unique solution (u,q) for the DPG formulation* (2.14). *Furthermore, the component q of the solution satisfies the boundary condition, i.e.,* $(B-M)(q-g) = 0$.

Proof. Lemma 2.4 implies that there exists a solution (u,q) for the DPG formulation (2.14) and the first component u is unique since it solves the strong equation (2.11). To prove the uniqueness of q, we first assume that (u,q_1) and (u,q_2) are two solutions of (2.14). Then, a simple subtraction shows that $(q_1-q_2) \in Q$, which in turns implies that $q_1 = q_2$ in the quotient space $\tilde{W}(\Omega)$. The last assertion is obvious from the last steps in the proof of Lemma 2.4. □

It should be pointed out that Corollary 2.5 provides the existence and uniqueness of the DPG solution for $f \in L(\Omega)$. In this case, the stability of the component u is ready due to the well-posedness of the strong problem (2.11). In order to obtain the well-posedness of the DPG formulation, the existence and uniqueness together with stability of both u and q must be established for all $f \in W'(\Omega_h)$. To this end, we first define the following norm

$$\|[\![v]\!]\|_{\partial\Omega_h} := \sup_{q\in\tilde{W}(\Omega)} \frac{a(q,v)}{\|q\|_{\tilde{W}}} = \sup_{r\in W(\Omega)} \frac{a(r,v)}{\|r\|_W} \tag{2.18}$$

for the "jump" $[\![v]\!]$, which will be clear for concrete examples in section 3. We next define norms in trial and test spaces such that both continuity and inf-sup constants are unity. One way to construct such norms is via a simple application of the Cauchy–Schwarz inequality to the bilinear form in (2.14) to have

$$b\left(\left(u,q\right),v\right) \le \underbrace{\left(\sum_{j=1}^{N^{\mathrm{el}}} \left\|u\right\|_{L(K_j)}^2 + \left\|q\right\|_{\tilde{W}(\Omega_h)}^2\right)^{\frac{1}{2}}}_{\left\|(u,q)\right\|_{opt}} \times \underbrace{\left(\sum_{j=1}^{N^{\mathrm{el}}} \left\|\tilde{T}v\right\|_{L(K_j)}^2 + \left\|[\![v]\!]\right\|_{\partial\Omega_h}^2\right)^{\frac{1}{2}}}_{\left\|v\right\|_{opt}},$$

where the subscript *opt* denotes the "natural optimal" norms in trial and test spaces correspondingly (see, e.g., [12] for a related approach). At this point, one needs to ensure that the optimal norm generates the same topology as that generated by the canonical norm in the broken graph space $W\left(\Omega_h\right)$. Here is the desired result.

THEOREM 2.6. *For all $v \in W\left(\Omega_h\right)$, there holds*

$$c_1 \left\|v\right\|_{opt} \le \left\|v\right\|_{W(\Omega_h)} \le c_2 \left\|v\right\|_{opt},$$

i.e., $\left\|\cdot\right\|_{opt}$ and $\left\|\cdot\right\|_{W(\Omega_h)}$ are equivalent, and hence generate the same topology in $W\left(\Omega_h\right)$.

Proof. Owing to the continuity of B, B_{K_j}, and M from (2.4) and (2.5), it is easy to see that $\left\|[\![v]\!]\right\|_{\partial\Omega_h} \le C\left\|v\right\|_{opt}$, and hence the optimal test norm is bounded from above by the broken graph norm. To obtain the converse, we adapt the argument proposed in [17] to our abstract framework. We begin by considering the following equation

$$(2.19) \qquad \begin{cases} \text{Given } v \in W\left(\Omega_h\right) \subset L\left(\Omega\right). \text{ Seek } w \in W\left(\Omega\right) \text{ such that} \\ Tw = v \text{ in } L\left(\Omega\right), \text{ and} \\ w \in V = \mathcal{N}\left(B - M\right). \end{cases}$$

By Theorem 2.3, (2.19) is well-posed and the following estimates hold:

$$\mu_0 \left\|w\right\|_{L(\Omega)} \le \left\|v\right\|_{L(\Omega)} \text{ and } \frac{\mu_0}{1+\mu_0}\left\|w\right\|_{W(\Omega)} \le \left\|v\right\|_{L(\Omega)}.$$

As a result, we have

$$\begin{aligned}
\left\|v\right\|_{L(\Omega)}^2 = \left(Tw, v\right)_{L(\Omega)} &= \sum_{j=1}^{N^{\mathrm{el}}} \left(w, \tilde{T}v\right)_{L(K_j)} + a\left(w,v\right) \\
&\le \sum_{j=1}^{N^{\mathrm{el}}} \left\|w\right\|_{L(K_j)} \left\|\tilde{T}v\right\|_{L(K_j)} + \sup_{r\in W(\Omega)} \frac{a\left(r,v\right)}{\left\|r\right\|_W} \left\|w\right\|_{W(\Omega)} \\
&\le \left(\sum_{j=1}^{N^{\mathrm{el}}} \left\|w\right\|_{L(K_j)}^2 + \left\|w\right\|_{W(\Omega)}^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{N^{\mathrm{el}}} \left\|\tilde{T}v\right\|_{L(K_j)}^2 + \left\|[\![v]\!]\right\|_{\partial\Omega_h}^2\right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{\left(1+\mu_0\right)^2}{\mu_0^2}} \left\|v\right\|_{L(\Omega)} \left\|v\right\|_{opt},
\end{aligned}$$

from which it follows that $\left\|v\right\|_{W(\Omega_h)} \le \sqrt{\frac{1+\mu_0^2+(1+\mu_0)^2}{\mu_0^2}} \left\|v\right\|_{opt}$. ∎

Let us define the DPG operator $\mathscr{B} : L(\Omega_h) \times \tilde{W}(\Omega) \to W'(\Omega_h)$ via the DPG bilinear form as

$$\langle \mathscr{B}(u,q), v\rangle_{W'(\Omega_h)\times W(\Omega_h)} := b((u,q), v).$$

We are now in position to discuss the well-posedness of the DPG formulation.

THEOREM 2.7 (well-posedness of the DPG formulation). *The DPG formulation* (2.14) *is well-posed, and the continuity and inf-sup constants are unity in the optimal norms. In particular,*

$$\|\mathscr{B}(u,q)\|_{opt} := \sup_{v\in W(\Omega_h)} \frac{b((u,q),v)}{\|v\|_{opt}} = \|(u,q)\|_{opt}.$$

Proof. Since, by the Riesz representation theorem, the equality in the Cauchy–Schwarz inequality $b((u,q),v) \leq \|(u,q)\|_{opt} \|v\|_{opt}$ is attainable, Theorem 2.6 in [8] shows that the continuity constant M and the inf–sup constant γ are unity. By the Banach–Nečas–Babuška theorem [20] (also known as the generalized Lax–Milgram theorem [4, 29]), the remaining task is to prove the following implication

$$\Big(b((u,q),v) = 0 \ \forall (u,q) \in L(\Omega_h) \times \tilde{W}(\Omega)\Big) \Rightarrow v = 0.$$

To this end, take $q = u \in V = \mathcal{N}(M - B) \subset W(\Omega)$. Then, applying (2.4) element by element the expression $b((u,q),v) = 0$ becomes

$$(Tu, v)_{L(\Omega)} = 0,$$

which yields $v = 0$ in $L(\Omega)$ since T is isomorphic from V to L as in Theorem 2.3, and this ends the proof. ∎

COROLLARY 2.8. *There holds*

$$\|\mathscr{B}(u,q)\|_{W'(\Omega_h)} := \sup_{v\in W(\Omega_h)} \frac{b((u,q),v)}{\|v\|_{W(\Omega_h)}} \geq \sqrt{\frac{\mu_0^2}{1+\mu_0^2+(1+\mu_0)^2}}\,\|(u,q)\|_{opt}.$$

Proof. Using Theorem 2.7 and the inequality at the end of the proof of Theorem 2.6 we have

$$\|(u,q)\|_{opt} = \sup_{v\in W(\Omega_h)} \frac{b((u,q),v)}{\|v\|_{opt}} \leq \sqrt{\frac{1+\mu_0^2+(1+\mu_0)^2}{\mu_0^2}} \sup_{v\in W(\Omega_h)} \frac{b((u,q),v)}{\|v\|_{W(\Omega_h)}},$$

which ends the proof. ∎

Remark 2.9. The result in Corollary 2.8 shows that the inf-sup constant of the DPG formulation (2.14) is independent of the mesh and the same order of the inf-sup constant μ_0 of the strong form (2.11). Note that, up to this point, the infinite dimensional DPG formulation (2.14) and its well-posedness are valid for any DPG method using L^2 as the trial space and (broken) graph space as the test space.

For the rest of the paper, for convenience, we denote the broken graph norm as $\|v\|_{lopt} := \|v\|_{W(\Omega_h)}$ and define the corresponding norm in the trial space

$$\|(u,q)\|_{lopt} := \sup_{v\in W(\Omega_h)} \frac{b((u,q),v)}{\|v\|_{lopt}} = \|\mathscr{B}(u,q)\|_{W'(\Omega_h)}.$$

We also denote the DPG method with optimal test norm as DPGopt and with broken graph test norm as DPGlopt. Note that they are different methods as shown in section 2.4.

2.4. Convergence of finite dimensional DPG methods with optimal test functions. Let us now denote $\mathcal{U} = L(\Omega_h) \times \tilde{W}(\Omega)$ and $\mathcal{V} = W(\Omega_h)$. Given a set of N independent basis functions $\{\varphi_i\}_{i=1}^N$ in the trial space $\mathcal{U}$, the corresponding optimal test functions $\psi_i = S\varphi_i \in \mathcal{V}, i = 1, \dots, N$, images of the *trial-to-test* operator S [14], can be computed by solving the following equation

$$(\psi_i, v)_{\mathcal{V}} = b(\varphi_i, v), \quad \forall v \in \mathcal{V}, \tag{2.20}$$

where $\|\cdot\|_{\mathcal{V}} \in \{\|\cdot\|_{opt}, \|\cdot\|_{lopt}\}$ is a norm in $\mathcal{V}$. Since our DPG formulation (2.14) is well-posed as proved in Theorem 2.7, S is bijective and hence $\{\psi_i\}_{i=1}^N$ is also a set of N independent basis functions in $\mathcal{V}$. Let us denote $\mathcal{U}_N = \operatorname{span}\{\varphi_i\}_{i=1}^N$, $\mathcal{V}_N = \operatorname{span}\{\psi_i\}_{i=1}^N$, and let (u_N, q_N) be the solution of

$$\begin{cases} \text{Seek } (u_N, q_N) \in \mathcal{U}_N \text{ such that} \\ b((u_N, q_N), v) = \ell(v) \quad \forall v \in \mathcal{V}_N. \end{cases} \tag{2.21}$$

Note that the well-posedness of this discrete equation is inherited from the continuous setting (2.14); see [8, 14, 15] for the detailed exposition. Then the following convergence result is standard.

THEOREM 2.10. *Let $\|\cdot\|_X, \|\cdot\|_Y \in \{\|\cdot\|_{opt}, \|\cdot\|_{lopt}\}$ be two norms in $\mathcal{V}$ such that*

$$c_1 \|v\|_X \le \|v\|_Y \le c_2 \|v\|_X \quad \forall v \in \mathcal{V}.$$

If the test basis functions $\{\psi_i\}_{i=1}^N$ are computed using the $\|\cdot\|_Y$-norm for the test space $\mathcal{V}$, then

$$\|(u,q) - (u_N, q_N)\|_X \le \frac{c_2}{c_1} \inf_{(w,p)\in\mathcal{U}_N} \|(u,q) - (w,p)\|_X .$$

Proof. See [30] for a proof. ∎

Clearly the error is optimal if we use the $\|\cdot\|_X$-norm for the test space $\mathcal{V}$ to compute the test basis functions. Furthermore, the stiffness matrix of the discrete problem is always symmetric positive definite. Indeed, the symmetry and the positive definiteness are direct consequences of the inner product in $\mathcal{V}$, i.e.,

$$b(\varphi_i, S\varphi_j) = (S\varphi_i, S\varphi_j)_{\mathcal{V}} = (S\varphi_j, S\varphi_i)_{\mathcal{V}} = b(\varphi_j, S\varphi_i).$$

Remark 2.11. Note that the broken graph norm $\|v\|_{lopt}$ allows one to compute optimal test functions elementwise, and hence practical, as opposed to the optimal test norm $\|v\|_{opt}$ that requires us to solve for each optimal test function on the whole mesh.

Remark 2.12. It is assumed that we can solve for the optimal test basis functions $\{\psi_i\}_{i=1}^N$ exactly. In practice, we approximate ψ_i by $\tilde{S}\varphi_i$, where $\tilde{S}$ is an approximation of S [26], namely, we replace (2.20) by

$$\left(\tilde{S}\varphi_i, v\right)_{\mathcal{V}} = b(\varphi_i, v) \quad \forall v \in \mathcal{V}_r \subset \mathcal{V}.$$

As a consequence, the discrete well-posedness is no longer inherited from the continuous one as in the ideal DPG methods. Nevertheless, under some suitable conditions, DPG methods with approximate optimal test functions are still well-posed [26].

3. Examples. For each set of PDEs considered in this section, we first convert the governing equations to a first order system (if necessary), followed by a trace theorem on a single domain (if available). Our task is to first provide the detailed and explicit specifications of B, M, V, and V^* for each system of PDEs. We then discuss a continuous extension of $(M-B)\,q$ from $W'(\Omega)$ to $W'(\Omega_h)$ along with the compatibility condition (2.13) and the space Q. These abstract objects, which may seem to be confusing on the abstract level at the first sight, become familiar entities adapting to each set of PDEs. One of the main results of our analysis is the equivalence of a vector in the quotient space, $q \in W/Q\,(\Omega)$, and its trace on the skeleton, thus making our DPG formulation practical. More important, the well-posedness result in section 2.3, which was developed for DPG formulation with $q \in W/Q\,(\Omega)$, can now be transfered to practical DPG formulation with traces. Finally, the DPG formulation specialized to the corresponding PDE is presented, followed by a discussion on the relation of our DPG methods and the existing ones in the literature. As will be shown, we recover several existing DPG methods and discover new ones for not only the PDEs that have been already studied but also those that have not been tackled by the DPG community.

We denote the skeleton of the mesh by $\Gamma_h = \cup_{j=1}^{N^{\rm el}} \partial K_j$, the set of all (uniquely defined) faces/edges e, each of which comes with a normal vector $\mathbf{n}_e$. The internal skeleton is then defined as $\Gamma_h^0 = \Gamma_h \setminus \partial\Omega$. If a face/edge $e \in \Gamma_h$ is the intersection of ∂K_i and ∂K_j, $i \neq j$, we define the following jumps:

$$[\![v]\!] := \operatorname{sgn}\left(\mathbf{n}^-\right) v^- + \operatorname{sgn}\left(\mathbf{n}^+\right) v^+, \quad [\![\boldsymbol{\tau}]\!] := \mathbf{n}^- \cdot \boldsymbol{\tau}^- + \mathbf{n}^+ \cdot \boldsymbol{\tau}^+,$$

where

$$\operatorname{sgn}\left(\mathbf{n}^\pm\right) := \begin{cases} 1 & \text{if } \mathbf{n}^\pm = \mathbf{n}_e \\ -1 & \text{if } \mathbf{n}^\pm = -\mathbf{n}_e. \end{cases}$$

For e belonging to the domain boundary $\partial\Omega$, we define

$$[\![v]\!] := v, \quad [\![\boldsymbol{\tau}]\!] := \mathbf{n}_e \cdot \boldsymbol{\tau} \quad \text{on } \partial\Omega.$$

Note that we allow the arbitrariness in assigning "−" and "+" quantities to the adjacent elements K_i and K_j.

For the rest of the paper, we use the same notation for both a function and its trace (if it is well defined) when there is no ambiguity.

3.1. Scalar advection-reaction equations. We consider the following scalar hyperbolic PDE (a related Petrov–Galerkin method for this equation can be found in [12]) over a Lipschitz domain Ω:

$$\boldsymbol{\beta} \cdot \nabla u + \mu u = f \quad \text{in } \Omega, \text{ and } u = g \quad \text{on } \partial\Omega^-,$$

where $\partial\Omega^+ := \{\mathbf{x} \in \partial\Omega : \boldsymbol{\beta} \cdot \mathbf{n} \geq 0\}$, $\partial\Omega^- := \{\mathbf{x} \in \partial\Omega : \boldsymbol{\beta} \cdot \mathbf{n} < 0\}$, $\boldsymbol{\beta} \in \left[W^{1,\infty}(\Omega)\right]^d$, $\mu \in L^\infty(\Omega)$, and

$$g \in L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}\left(\partial\Omega^-\right) := \left\{ v : \|v\|^2_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\partial\Omega^-)} = \int_{\partial\Omega^-} |\boldsymbol{\beta} \cdot \mathbf{n}|\, |v|^2 \, ds < \infty \right\}.$$

For convenience in writing, we also define $\Gamma_h^{\pm} := \Gamma_h \setminus \partial\Omega^{\mp}$. We assume there exists $\mu_0 > 0$ such that

$$\mu - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \geq \mu_0 > 0 \quad \text{a.e in } \Omega. \tag{3.1}$$

Note that assumption (3.1) is not a limitation since it is always valid under a change of variable with exponential factor [25, 27]. Clearly, the graph space is given by

$$W(\Omega) = \left\{u \in L^2(\Omega) : \boldsymbol{\beta} \cdot \nabla u \in L^2(\Omega)\right\} =: H^1_{\boldsymbol{\beta}}(\Omega).$$

This is a particular instance of Friedrichs' systems considered in section 2.2 with $m = 1$, $C = \mu$, and $A^k = \beta_k$, where β_k is the kth component of vector $\boldsymbol{\beta}$. The following proposition summarizes some of the results on the trace, M, B, V, and V^* in [21, 24] for a single domain.

LEMMA 3.1. *Assume that $\partial\Omega^-$ and $\partial\Omega^+$ are well-separated, i.e., dist$(\partial\Omega^-, \partial\Omega^+) > 0$. Then the following hold.*

(i) *The trace operator $\gamma : H^1_{\boldsymbol{\beta}}(\Omega) \to L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\partial\Omega)$ is a continuous surjection.*

(ii) *$\mathcal{B} = \boldsymbol{\beta} \cdot \mathbf{n}$ and the boundary operator B satisfies*

$$\langle Bu, v\rangle_{W'(\Omega)\times W(\Omega)} = \int_{\partial\Omega} \boldsymbol{\beta}\cdot\mathbf{n}uv\ ds \quad \forall u, v \in H^1_{\boldsymbol{\beta}}(\Omega).$$

(iii) *Define $\langle Mu, v\rangle_{W'(\Omega)\times W(\Omega)} = \int_{\partial\Omega} |\boldsymbol{\beta}\cdot\mathbf{n}|\, uv\, ds$; then M satisfies* (2.5a) *and* (2.5b). *Furthermore,*

$$V = \left\{v \in H^1_{\boldsymbol{\beta}}(\Omega) : \boldsymbol{\beta}\cdot\mathbf{n}\ v|_{\partial\Omega^-} = 0\right\}, \quad V^* = \left\{v \in H^1_{\boldsymbol{\beta}}(\Omega) : \boldsymbol{\beta}\cdot\mathbf{n}\ v|_{\partial\Omega^+} = 0\right\}.$$

What remains to be studied are the compatibility condition and the quotient space $\tilde{H}^1_{\boldsymbol{\beta}}(\Omega) = H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$. We assume that the mesh satisfies the separation condition in Lemma 3.1, namely, ∂K_j^- and ∂K_j^+ are well-separated[2] for all $j = 1, \ldots, N^{\text{el}}$. Without loss of generality, it is assumed that $\boldsymbol{\beta}\cdot\mathbf{n} \neq 0$ a.e. on Γ_h^+ in the following theorem since otherwise we can always redefine Γ_h^+ by taking away any nontrivial measure subsets of Γ_h^+ on which $\boldsymbol{\beta}\cdot\mathbf{n} = 0$, since they do not contribute to the DPG bilinear form. Using results of B, M and the trace operator in Lemma 3.1, a natural extension of $(M - B)\,q$ from $W'(\Omega)$ to $W'(\Omega_h)$ is specified as

$$\langle (M - B)\,q, v\rangle_{[H^1_{\boldsymbol{\beta}}(\Omega_h)]'\times H^1_{\boldsymbol{\beta}}(\Omega_h)} = -2\sum_{e\in\partial\Omega_h^-}\int_e \boldsymbol{\beta}\cdot\mathbf{n}qv\ ds$$

for any $q \in H^1_{\boldsymbol{\beta}}(\Omega)$ and $v \in H^1_{\boldsymbol{\beta}}(\Omega_h)$. Consequently, the compatibility condition (2.13) is trivial. We next study the quotient space $\tilde{H}^1_{\boldsymbol{\beta}}(\Omega) = H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ and its trace on the mesh skeleton.

[2]On the one hand, this is only a sufficient condition. On the other hand, as shown in [24], if this condition is violated the second assertion in Lemma 3.1 does not hold in general. In practice, this assumption is somewhat impractical even on a single domain, but we need it for the theory to go through rigorously.

THEOREM 3.2.

(i) $Q = \{q \in H^1_{\boldsymbol{\beta}}(\Omega) : q = 0 \text{ on } \Gamma_h^+\}$. *Furthermore,* $H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ *is isomorphic to* $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$. *In particular, the trace of a function in the quotient space* $H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ *is independent of its representations.*

(ii) *For each* $\hat{u} \in L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$, *define a new norm*

$$\|\hat{u}\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)} = \|[q]\|_{H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)},$$

where $[q] \in H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ *such that there exists a representation* q *satisfying* $q = \hat{u}$ *on* Γ_h^+. *Then,* $\| \cdot \|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)}$ *is equivalent to* $\|\cdot\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)}$, *and hence generating the same topology in* $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$. *In particular,* $H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ *is homeomorphic to* $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$.

Proof.

(i) The results in Lemma 3.1 allow us to write $a(q,v)$ as

$$a(q,v) = \int_{\Gamma_h^+} |\boldsymbol{\beta}\cdot\mathbf{n}|\, q[\![v]\!]\, ds = \sum_{e\in\Gamma_h^+} \int_e |\boldsymbol{\beta}\cdot\mathbf{n}|\, q[\![v]\!]\, ds,$$

and to conclude that $\gamma : H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega) \to L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$ is surjective. Clearly, $a(q,v) = 0\ \forall v \in H^1_{\boldsymbol{\beta}}(\Omega_h)$ implies that $\gamma q = 0$ on any subset of Γ_h^+, and hence the first assertion follows. The injectivity of γ can be shown as follows. Let $q_1, q_2 \in H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ such that their traces on Γ_h^+ are the same. Then one has

$$a(q_1 - q_2, v) = \sum_{e\in\Gamma_h^+} \int_e |\boldsymbol{\beta}\cdot\mathbf{n}|\,(q_1 - q_2)\,[\![v]\!]\, ds = 0 \quad \forall v \in H^1_{\boldsymbol{\beta}}(\Omega_h),$$

which implies $q_1 = q_2$ in $H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$.

(ii) The definition of the new norm is meaningful due to (i) and the definition of norm in the quotient space. Now, since $\gamma : q \mapsto \gamma q$ is a continuous surjection from $H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ to $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$, we have

$$\|\gamma q\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)} \le c_2 \|q\|_{H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)} = c_2 \|\gamma q\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)}.$$

On the other hand, since $H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ and $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$ are Banach spaces, and γ is bijective, a direct consequence of the Open Mapping theorem [29] shows that

$$\|\gamma q\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)} \ge c_1 \|q\|_{H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)} = c_1 \|\gamma q\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)}.$$

Thus, the equivalence of the norms and the homeomorphism follow. ∎

Remark 3.3. One can view the second assertion of Theorem 3.2 as an extension of the single domain trace theorem presented in the first assertion of Lemma 3.1 to the trace on the skeleton of the mesh. This is a natural task for us to do in order to explore the quotient space $H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ when the graph space has a well-defined trace space. Note that the trace theorem is typically established for a single domain with Lipschitz boundary [28]. Here, we need the trace on the skeleton Γ_h, for which the result on a single domain does not seem to be directly applicable, and Theorem 3.2 establishes such a result rigorously. More important, Theorem 3.2 casts our abstract DPG formulation to the usual form that can be implemented efficiently on a computer.

As a direct consequence of Theorem 3.2, we can identify $q \in H^1_{\boldsymbol{\beta}}(\Omega)/Q(\Omega)$ with $\hat{u} \in L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$, and we can use either $\|\cdot\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)}$ or $\|\cdot\|_{L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)}$ as the norm in $L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+)$. The ultraweak formulation (2.14) can now be written equivalently as

$$\text{Given } g \in L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\partial\Omega^-), f \in \left[H^1_{\boldsymbol{\beta}}(\Omega_h)\right]'. \text{ Seek } (u,\hat{u}) \in L(\Omega_h) \times L^2_{\boldsymbol{\beta}\cdot\mathbf{n}}(\Gamma_h^+) \text{ such that}$$

$$\sum_{j=1}^{N^{\mathrm{el}}} \int_{K_j} u\left(-\nabla\cdot(\boldsymbol{\beta}v) + \mu v\right)\, d\mathbf{x} + \sum_{e\in\Gamma_h^+} \int_e |\boldsymbol{\beta}\cdot\mathbf{n}|\, \hat{u}[\![v]\!]\, ds$$
$$= \langle f, v\rangle_{\left[H^1_{\boldsymbol{\beta}}(\Omega_h)\right]'\times H^1_{\boldsymbol{\beta}}(\Omega_h)} - \int_{e\in\partial\Omega^-} \boldsymbol{\beta}\cdot\mathbf{n} g v\, ds \quad \forall v \in H^1_{\boldsymbol{\beta}}(\Omega_h). \tag{3.2}$$

It follows that all the results in section 2 hold for (3.2). The DPGopt coincides with the second DPG method analyzed in [8] and the DPGlopt recovers the DPG method used, but not analyzed, in [15] for the two dimensional transport equation. The beauty of the abstract formulation here is that the well-posedness of both DPG methods is immediately available for transport equations in any dimensions.

3.2. Convection-diffusion-reaction equations. The problem of interest in this section is the convection-diffusion-reaction equation written in the first order form

$$\varepsilon^{-1}\boldsymbol{\sigma} + \nabla u = 0 \quad \text{in } \Omega, \tag{3.3a}$$

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{\beta}\cdot\nabla u + \mu u = f \quad \text{in } \Omega, \tag{3.3b}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.3c}$$

where we assume $\boldsymbol{\beta} \in [L^\infty(\Omega)]^d$, $\nabla\cdot\boldsymbol{\beta} \in L^\infty(\Omega)$, and ε is a $d\times d$ symmetric positive definite matrix with smallest eigenvalue uniformly bounded away from zero. We now relax condition (3.1) by the following weaker assumption

$$\operatorname*{ess\,inf}_{\Omega}\left(\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\right) \geq 0; \tag{3.4}$$

then it is trivial to see that condition (2.10c) holds. What remains to be checked is condition (2.10e), but this is immediate by the Poincaré inequality. Consequently, (3.3) is a particular instance of Friedrichs' system with partial coercivity introduced in section 2.2 with $m = d + 1$. It is also not difficult to see that the graph space is given by $W = H(\mathrm{div},\Omega)\times H^1(\Omega)$. The following proposition summarizes some of the results in [21] for a single domain.

LEMMA 3.4.

(i) *The trace operator*

$$\gamma : H(div,\Omega) \times H^1(\Omega) \ni (\boldsymbol{\sigma}, u) \mapsto (\boldsymbol{\sigma}\cdot\mathbf{n}, u) \in H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is a continuous surjection satisfying

$$\langle B(\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)\rangle_{W'(\Omega)\times W(\Omega)} = \langle \boldsymbol{\sigma}\cdot\mathbf{n}, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} + \langle \boldsymbol{\tau}\cdot\mathbf{n}, u\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} + \int_{\partial\Omega} \boldsymbol{\beta}\cdot\mathbf{n} uv \, ds.$$

(ii) *Define*

$$\langle M(\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)\rangle_{W'(\Omega)\times W(\Omega)} = \langle \boldsymbol{\sigma}\cdot\mathbf{n}, v\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} - \langle \boldsymbol{\tau}\cdot\mathbf{n}, u\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)};$$

then M satisfies (2.5a) *and* (2.5b). *Furthermore,*

$$V = V^* = \left\{(\boldsymbol{\sigma}, u) \in H(div,\Omega) \times H^1(\Omega) : u|_{\partial\Omega} = 0\right\} = H(div,\Omega) \times H^1_0(\Omega).$$

For any $q = (q^{\boldsymbol{\sigma}}, q^u) \in W(\Omega)$, Lemma 3.6 suggests that a natural extension of $(M-B)\,q$ from $W'(\Omega)$ to $W'(\Omega_h)$ be given by

$$\langle (M-B)\,q, (\boldsymbol{\tau}, v)\rangle_{W'(\Omega_h)\times W(\Omega_h)} = -2\,\langle \boldsymbol{\tau}\cdot\mathbf{n}, q^u\rangle_{H^{-\frac{1}{2}}(\partial\Omega_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega_h;\mathbb{R}^d)} - \int_{\partial\Omega_h} \boldsymbol{\beta}\cdot\mathbf{n} q^u v \, ds,$$

from which the compatibility condition (2.13) is trivial.

As shown in [21], the boundary matrix M is not unique. In fact there are infinitely many of them, and our choice is probably the simplest. Next, we study the quotient space $\tilde{W}(\Omega) = (H(\mathrm{div},\Omega) \times H^1(\Omega))/Q(\Omega)$. As in section 3.1, we assume that if $\boldsymbol{\beta}$ is not identically zero then $\boldsymbol{\beta}\cdot\mathbf{n} \neq 0$ a.e. on Γ_h. Here is a result parallel to Theorem 3.2.

THEOREM 3.5.

(i) *The subspace Q is given by*

$$Q = \left\{ q \in H(div,\Omega) \times H^1(\Omega) : (q^{\boldsymbol{\sigma}}\cdot\mathbf{n}, q^u) = 0 \text{ on } \Gamma_h^0 \text{ and } q^{\boldsymbol{\sigma}}\cdot\mathbf{n} = -\frac{1}{2}\,|\boldsymbol{\beta}\cdot\mathbf{n}|\, q^u \text{ in } H^{-\frac{1}{2}}(\partial\Omega_h) \right\}.$$

Furthermore, $(H(div,\Omega)\times H^1(\Omega))/Q(\Omega)$ is isomorphic to $H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)$. In particular, the trace of a function in the quotient space $(H(div,\Omega)\times H^1(\Omega))/Q(\Omega)$ is independent of its representations.

(ii) *For each $(\hat{\boldsymbol{\sigma}}, \hat{u}) \in H^{-\frac{1}{2}}(\Gamma_h) \times H^{\frac{1}{2}}(\Gamma_h)$, define a new norm*

$$\|\!|(\hat{\boldsymbol{\sigma}}, \hat{u})\|\!|_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)} = \|[q]\|_{(H(div,\Omega)\times H^1(\Omega))/Q(\Omega)},$$

where $[q] \in (H(div,\Omega)\times H^1(\Omega))/Q(\Omega)$ such that there exists a presentation q of $[q]$ satisfying $\gamma q = (q^{\boldsymbol{\sigma}}\cdot\mathbf{n}, q^u) = (\hat{\boldsymbol{\sigma}}, \hat{u})$ on Γ_h. Then, $\|\!|\cdot\|\!|_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}$ is equivalent to $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}$, and hence generating the same topology in $H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)$. In particular, $(H(div,\Omega)\times H^1(\Omega))/Q(\Omega)$ and $H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)$ are homeomorphic.

Proof. For this example, one has

$$a\left(q,(\boldsymbol{\tau},v)\right) = \sum_{j=1}^{N^{\text{el}}} \frac{1}{2}\int_{\partial K_j} \left|\boldsymbol{\beta}\cdot\mathbf{n}\right| q^u [\![v]\!] \; ds + \left\langle [\![\boldsymbol{\tau}]\!], q^u \right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right)}$$
$$+ \sum_{e\in\Gamma_h} \left\langle q^{\boldsymbol{\sigma}}\cdot\mathbf{n}_e, [\![v]\!]\right\rangle_{H^{-\frac{1}{2}}(e)\times H^{\frac{1}{2}}(e)}.$$

The surjectivity of the trace operator allows us to easily show that $a\left(q,(\boldsymbol{\tau},v)\right)=0$ $\forall\left(\boldsymbol{\tau},v\right)\in H\left(\text{div},\Omega_h\right)\times H^1\left(\Omega_h\right)$ implies $q=\left(q^{\boldsymbol{\sigma}}\cdot\mathbf{n},q^u\right)=0$ on Γ_h^0 and $q^{\boldsymbol{\sigma}}\cdot\mathbf{n},= -\frac{1}{2}\boldsymbol{\beta}\cdot\mathbf{n}q^u$ on $H^{-\frac{1}{2}}\left(\Omega_h\right)$. Indeed, take $v=0$, then it can be deduced that $q^u=0$ on Γ_h^0. Next, take $v\in H_0^1\left(\Omega_h\right)$, we infer that $q^{\boldsymbol{\sigma}}=0$ on Γ_h^0, and the second assertion now follows. The rest of the proof is similar to that of Theorem 3.2. ∎

As a direct consequence of Theorem 3.5, we can identify $q\in(H(\text{div},\Omega)\times H^1(\Omega))/Q(\Omega)$ with $(\hat{\boldsymbol{\sigma}},\hat{u})\in H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)$, and we can use either $[\![\cdot]\!]_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}$ or $\left\|\cdot\right\|_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}$ as the norm in $H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)$. The abstract DPG formulation (2.14) now equivalently becomes:

Given $f\in\left[H^1\left(\Omega_h\right)\right]'$.

Seek $(\boldsymbol{\sigma},u,\hat{\boldsymbol{\sigma}},\hat{u})\in\left[L^2\left(\Omega_h\right)\right]^d\times L^2\left(\Omega_h\right)\times H^{-\frac{1}{2}}\left(\Gamma_h\right)\times H^{\frac{1}{2}}\left(\Gamma_h\right)$ such that

$$\sum_{j=1}^{N^{\text{el}}}\int_{K_j} \boldsymbol{\sigma}\cdot\left(\varepsilon^{-1}\boldsymbol{\tau}-\nabla v\right)+u\left(-\nabla\cdot\boldsymbol{\tau}-\nabla\cdot(\boldsymbol{\beta}v)+\mu v\right)\; d\mathbf{x}+\frac{1}{2}\int_{\partial K_j}\left|\boldsymbol{\beta}\cdot\mathbf{n}\right|\hat{u}[\![v]\!]\; ds$$
$$+\left\langle[\![\boldsymbol{\tau}]\!],\hat{u}\right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right)}+\left\langle\hat{\boldsymbol{\sigma}},[\![v]\!]\right\rangle_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)}$$
$$=\left\langle f,v\right\rangle_{[H^1(\Omega_h)]'\times[H^1(\Omega_h)]}\quad\forall\left(\boldsymbol{\tau},v\right)\in H\left(\text{div},\Omega_h\right)\times H^1\left(\Omega_h\right). \tag{3.5}$$

Consequently, results in section 2 are valid for (3.5). More specifically, the well-posedness of DPGopt and DPGlopt is readily available for (3.5). It should be pointed out that the DPGopt and DPGlopt for (3.5), with $\boldsymbol{\beta}=\mathbf{0}$ and $\mu=0$, are identical to those analyzed in [14] for the Poisson equation if $f\in L^2\left(\Omega\right)$. Here, our approach is novel in the sense that the function spaces and the well-posedness of the corresponding DPG formulation are the direct consequences of the single abstract framework developed in section 2 for all $f\in\left[H^1\left(\Omega_h\right)\right]'\supset L^2\left(\Omega\right)$. However, we admit the fact that taking advantage of the particular structure of a PDE under consideration may yield sharper stability estimates and much more. This is not possible for our abstract and unified framework in section 2.

It turns out that the DPGlopt coincides with the DPG method used in [15] for the convection-diffusion problem ($\mu=0$) in two dimensions. (Actually, there is a slight difference in imposing the boundary condition for the convection term, i.e, the third term on the right side of (3.5); we have a factor 1/2 at the domain boundary $\partial\Omega_h$ instead of 1 as in [15].) However, while the DPGlopt method is assumed to be well-posed in [15], our results in section 2 show that it is indeed the case and the proof is the direct consequence of Theorem 2.7. Moreover, our function space setting for $\hat{u}$ comes out naturally from the abstract setting as the trace of the graph space while it is left unspecified in [15]. Recently, the authors of [15] have analyzed their

DPG method for the convection-diffusion problem in [14] where they combine the diffusion flux $\hat{\boldsymbol{\sigma}}$ and convection flux $|\boldsymbol{\beta} \cdot \mathbf{n}|\, \hat{u}$ into a single unknown total flux. This again comes out naturally from our abstract DPG, and therefore we recover the DPG method in [14]. Nevertheless, our abstract framework is not able to recover the robust versions of the DPG method developed in [19].

3.3. Linear(ized) continuum mechanics. The problem of interest in this section is governed by

$$
\begin{aligned}
\mathcal{A}\boldsymbol{\sigma} - \tfrac{1}{2}\left(\nabla u + (\nabla u)^T\right) &= 0 \quad \text{in } \Omega,\\
-\tfrac{1}{2}\nabla \cdot \left(\boldsymbol{\sigma} + \boldsymbol{\sigma}^T\right) + \boldsymbol{\beta} \cdot \nabla u + \mu u &= f \quad \text{in } \Omega,\\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\tag{3.6}
$$

where $\mathcal{A}$ is the compliance tensor, u the displacement in solid mechanics or velocity in fluid mechanics, and $\boldsymbol{\sigma}$ the stress tensor. Note that the stress tensor $\boldsymbol{\sigma}$ with values in $\mathbb{R}^{d,d}$ can be identified with a vector-valued field in $\mathbb{R}^{d^2}$. However, to simplify the notations we use the same symbol $\boldsymbol{\sigma}$ for both tensor-valued and vector-valued fields, and this should be clear in each context. Similarly, we identify the tensor $\mathcal{A}$ with a matrix in $\mathbb{R}^{d^2,d^2}$.

Assume that $\mathcal{A}$ is self-adjoint and uniformly positive definite on $\mathbb{R}^{d,d}$ with each component in $L^\infty(\Omega)$. We further assume that (3.4) holds. Set $m = d^2 + d$, $m_{\boldsymbol{\sigma}} = d^2$, and $m_u = d$. Thus, the full coercivity (3.1) does not hold, but the partial coercivity (2.10c) does. It is straightforward to cast (3.6) into the framework of a two-field Friedrichs' system in section 2.2.2. Indeed, (3.6) satisfies hypotheses (2.7a), (2.7b), and (2.7c) if $\boldsymbol{\beta} \in \left[W^{1,\infty}(\Omega)\right]^d$ and $\mu \in L^\infty(\Omega)$. In general, (2.9) does not hold unless $\mu_0 > 0$. Fortunately, (2.10c) holds since $C^{\boldsymbol{\sigma}\boldsymbol{\sigma}} = \mathcal{A}$ is uniformly positive definite. What remains to be checked is the assumption (2.10e), but this is clear by Korn's first inequality. Thus, (3.6) fulfills all the conditions of the two-field Friedrichs' system discussed in section 2.2.2.

Let us denote $H(div, \Omega; \mathbb{R}^{d,d}) = \{\boldsymbol{\sigma} \in L^2(\Omega; \mathbb{R}^{d,d}) : \nabla \cdot (\boldsymbol{\sigma} + \boldsymbol{\sigma}^T) \in L^2(\Omega; \mathbb{R}^d)\}$, where the divergence operator acts rowwise. Then, the graph space [22, 23] is given by $W(\Omega) = H(div, \Omega; \mathbb{R}^{d,d}) \times H^1(\Omega; \mathbb{R}^d)$. Next, we extract from [23] the properties of B, M, V, V^*, and the trace operator γ for a single domain.

LEMMA 3.6. *The following hold.*

(i) *The trace operator* γ *defined by*

$$
\gamma : H\left(div, \Omega; \mathbb{R}^{d,d}\right) \times H^1\left(\Omega; \mathbb{R}^d\right) \to H^{-\frac{1}{2}}\left(\partial\Omega; \mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\partial\Omega; \mathbb{R}^d\right),
$$
$$
(\boldsymbol{\sigma}, u) \overset{\gamma}{\mapsto} (\boldsymbol{\sigma} \cdot \mathbf{n}, u)
$$

is a continuous surjection satisfying

$$
\begin{aligned}
&\langle B(\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)\rangle_{W'(\Omega)\times W(\Omega)}\\
&\quad = -\left\langle \frac{1}{2}\left(\boldsymbol{\sigma} + \boldsymbol{\sigma}^T\right) \cdot \mathbf{n}, v \right\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)}\\
&\qquad - \left\langle \frac{1}{2}\left(\boldsymbol{\tau} + \boldsymbol{\tau}^T\right) \cdot \mathbf{n}, u \right\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)} + \int_{\partial\Omega} \boldsymbol{\beta} \cdot \mathbf{n} uv \, ds.
\end{aligned}
$$

(ii) *Define*

$$\langle M(\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)\rangle_{W'(\Omega)\times W(\Omega)} = -\left\langle \frac{1}{2}\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^T\right)\cdot \mathbf{n}, v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)} + \left\langle \frac{1}{2}\left(\boldsymbol{\tau}+\boldsymbol{\tau}^T\right)\cdot \mathbf{n}, u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^d)};$$

then M satisfies (2.5a) *and* (2.5b)*. Furthermore,*

$$V = V^* = H\left(div, \Omega; \mathbb{R}^{d,d}\right)\times H_0^1\left(\Omega;\mathbb{R}^d\right).$$

For any $q = (q^{\boldsymbol{\sigma}}, q^u) \in W(\Omega)$, Lemma 3.6 suggests that a natural extension of $(M-B)\,q$ from $W'(\Omega)$ to $W'(\Omega_h)$ be given by

$$\langle (M-B)\,q, (\boldsymbol{\tau}, v)\rangle_{W'(\Omega_h)\times W(\Omega_h)} = \left\langle \left(\boldsymbol{\tau}+\boldsymbol{\tau}^T\right)\cdot\mathbf{n}, q^u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\partial\Omega_h;\mathbb{R}^d)} - \int_{\partial\Omega_h} \boldsymbol{\beta}\cdot\mathbf{n} q^u v\, ds,$$

from which the compatibility condition (2.13) is trivial.

Next, we study the quotient space $\tilde{W}(\Omega) = (H(\mathrm{div}, \Omega; \mathbb{R}^{d,d}) \times H^1(\Omega;\mathbb{R}^d))/Q(\Omega)$. As in section 3.1, we assume that if $\boldsymbol{\beta}$ is not identically zero then $\boldsymbol{\beta}\cdot\mathbf{n} \neq 0$ a.e. on Γ_h. Here is a result parallel to Theorem 3.2.

THEOREM 3.7.

(i) *The subspace Q is given by*

$$Q = \left\{ q \in H\left(div, \Omega; \mathbb{R}^{d,d}\right)\times H^1\left(\Omega;\mathbb{R}^d\right) : \left(\frac{1}{2}\left(q^{\boldsymbol{\sigma}} + (q^{\boldsymbol{\sigma}})^T\right)\cdot\mathbf{n}, q^u\right) = 0 \text{ on } \Gamma_h^0 \text{ and } \left(q^{\boldsymbol{\sigma}} + (q^{\boldsymbol{\sigma}})^T\right)\cdot\mathbf{n} = |\boldsymbol{\beta}\cdot\mathbf{n}|\, q^u \text{ in } H^{-\frac{1}{2}}\left(\partial\Omega_h;\mathbb{R}^d\right)\right\}.$$

Furthermore, $(H(div, \Omega; \mathbb{R}^{d,d}) \times H^1(\Omega;\mathbb{R}^d))/Q(\Omega)$ is isomorphic to $H^{-\frac{1}{2}}(\Gamma_h; \mathbb{R}^d) \times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)$. In particular, the trace of a function in the quotient space $(H(div, \Omega; \mathbb{R}^{d,d}) \times H^1(\Omega;\mathbb{R}^d))/Q(\Omega)$ is independent of its representations.

(ii) *For each $(\hat{\boldsymbol{\sigma}}, \hat{u}) \in H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)$, define a new norm*

$$\left\| (\hat{\boldsymbol{\sigma}}, \hat{u}) \right\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)} = \|[q]\|_{(H(div,\Omega;\mathbb{R}^{d,d})\times H^1(\Omega;\mathbb{R}^d))/Q(\Omega)},$$

where $[q] \in (H(div, \Omega; \mathbb{R}^{d,d}) \times H^1(\Omega;\mathbb{R}^d))/Q(\Omega)$ is such that there exists a representation q satisfying $\gamma q = (\frac{1}{2}(q^{\boldsymbol{\sigma}} + (q^{\boldsymbol{\sigma}})^T)\cdot\mathbf{n}, q^u) = (\hat{\boldsymbol{\sigma}}, \hat{u})$ on Γ_h. Then, $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)}$ is equivalent to $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)}$, and hence generating the same topology in $H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)$. In particular, $(H(div, \Omega; \mathbb{R}^{d,d}) \times H^1(\Omega;\mathbb{R}^d))/Q(\Omega)$ and $H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)\times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)$ are homeomorphic.

Proof. For this example, one has

$$a(q, (\boldsymbol{\tau}, v)) = \sum_{j=1}^{N^{\mathrm{el}}} \frac{1}{2}\int_{\partial K_j} |\boldsymbol{\beta}\cdot\mathbf{n}|\, q^u [\![v]\!]\, ds - \left\langle \frac{1}{2}[\![\boldsymbol{\tau}+\boldsymbol{\tau}^T]\!], q^u\right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)\times H^{\frac{1}{2}}\left(\Gamma_h^0\right);\mathbb{R}^d} - \sum_{e\in\Gamma_h}\left\langle \frac{1}{2}\left(q^{\boldsymbol{\sigma}} + (q^{\boldsymbol{\sigma}})^T\right)\cdot\mathbf{n}_e, [\![v]\!]\right\rangle_{H^{-\frac{1}{2}}(e;\mathbb{R}^d)\times H^{\frac{1}{2}}(e;\mathbb{R}^d)}.$$

The rest of the proof is similar to that of Theorem 3.5. ∎

Theorem 3.7 suggests that we can identify $q \in (H(\mathrm{div},\Omega;\mathbb{R}^{d,d}) \times H^1(\Omega;\mathbb{R}^d))/Q(\Omega)$ with $(\hat{\boldsymbol{\sigma}},\hat{u}) \in H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d) \times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)$. The abstract DPG formulation (2.14) now equivalently becomes

$$
\begin{aligned}
&\text{Given } f \in \left[H^1\left(\Omega_h;\mathbb{R}^d\right)\right]'.\\
&\text{Seek } (\boldsymbol{\sigma},u,\hat{\boldsymbol{\sigma}},\hat{u}) \in L^2\left(\Omega_h;\mathbb{R}^{d,d}\right) \times L^2\left(\Omega_h;\mathbb{R}^d\right) \times H^{-\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h;\mathbb{R}^d\right)\\
&\text{such that, } \forall (\boldsymbol{\tau},v) \in H\left(\mathrm{div},\Omega_h;\mathbb{R}^{d,d}\right) \times H^1\left(\Omega_h;\mathbb{R}^d\right),\\
&\sum_{j=1}^{N^{\mathrm{el}}} \int_{K_j} \boldsymbol{\sigma} : \left(\mathcal{A}\boldsymbol{\tau} + \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)\right) + u \cdot \left(\frac{1}{2}\nabla \cdot \left(\boldsymbol{\tau} + \boldsymbol{\tau}^T\right) - \nabla \cdot (\boldsymbol{\beta} v) + \mu v\right) \, d\mathbf{x}\\
&\qquad + \sum_{j=1}^{N^{\mathrm{el}}} \frac{1}{2} \int_{\partial K_j} \boldsymbol{\beta} \cdot \mathbf{n}\hat{u} \cdot [\![v]\!] \, ds - \left\langle \frac{1}{2}[\![\boldsymbol{\tau} + \boldsymbol{\tau}^T]\!], \hat{u} \right\rangle_{H^{-\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right) \times H^{\frac{1}{2}}\left(\Gamma_h^0;\mathbb{R}^d\right)}\\
&\qquad - \langle \hat{\boldsymbol{\sigma}}, [\![v]\!] \rangle_{H^{-\frac{1}{2}}(\Gamma_h;\mathbb{R}^d) \times H^{\frac{1}{2}}(\Gamma_h;\mathbb{R}^d)} = \langle f, v \rangle_{[H^1(\Omega_h;\mathbb{R}^d)]' \times H^1(\Omega_h;\mathbb{R}^d)} .
\end{aligned}
\tag{3.7}
$$

Consequently, the results in section 2 hold. The DPGlopt for linear elasticity equations ($\boldsymbol{\beta} = \mathbf{0}$ and $\mu = 0$) is related to the DPG method analyzed in [7], but here in this paper our well-posedness proof is different and comes directly from section 2. A linearized version of the compressible Navier–Stokes equations considered in [23] is corresponding to $\mathcal{A} = I_{d^2} - \frac{1}{d+\lambda}\mathcal{Z}$, where $\lambda > 0$ is the compressibility factor, and $\mathcal{Z}_{[ij][kl]} = \delta_{ij}\delta_{kl}$. Compared to the existing DPG method for one dimensional Navier–Stokes equation in [10], our two DPG methods seem to be the first efforts in developing DPG approaches with guaranteed well-posedness to a multidimensional linearized version of the Navier–Stokes equations.

3.4. Time-domain acoustic equations. In this section, we apply our abstract framework devised in section 2 to time-domain acoustic equations. Alternatively, one can consider frequency-domain acoustic equations leading to Helmholtz equations for which a DPG method has been proposed and analyzed in [18]. The time-domain acoustic equations in the pressure-velocity form are given by

$$
\begin{aligned}
\rho c^2 \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \times (0,T_f),\\
\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0,T_f),\\
p(\mathbf{x},0) = p_0(\mathbf{x}), \quad \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega,\\
\mathbf{u} \cdot \mathbf{n} &= \lambda p && \text{in } \partial\Omega \times (0,T_f),
\end{aligned}
$$

where ρ is the density, c the speed of sound, p the pressure, and $\mathbf{u}$ the velocity vector. There are several approaches to deal with time-dependent problems. For example, one can use our DPG framework simultaneously for both space and time to arrive at a space-time DPG formulation (see Chan, Demkowicz, and Shashkov [11] for a space-time DPG formulation of one dimensional convection, convection-diffusion, and Burger's equations). Here, we explore a simple approach to cast the time-dependent acoustic equations into a Friedrichs' system discussed in section 2.2. To this end, we first assume that both $\rho \in L^\infty(\Omega)$ and $c \in L^\infty$ are positive and uniformly bounded away from zero. Next, we discretize the time derivative, using the backward Euler method for example, to arrive at the following generic equations for each time step:

$$\begin{aligned}
\varepsilon\boldsymbol{\sigma} + \nabla u &= \mathbf{f} && \text{in } \Omega,\\
\mu u + \nabla\cdot\boldsymbol{\sigma} &= g && \text{in } \Omega,\\
\boldsymbol{\sigma}\cdot\mathbf{n} &= \lambda u && \text{on } \partial\Omega,
\end{aligned}$$

which, similar to section 3.2, is clearly a Friedrichs' system discussed in section 2.2 with $m = d+1$. Similar to section 3.2, the graph space is

$$W = H\left(\mathrm{div},\Omega\right)\times H^1\left(\Omega\right)$$

and

$$\begin{aligned}
\left\langle B\left(\boldsymbol{\sigma},u\right),\left(\boldsymbol{\tau},v\right)\right\rangle_{W'(\Omega)\times W(\Omega)} &= \left\langle \boldsymbol{\sigma}\cdot\mathbf{n},v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}\\
&\quad + \left\langle \boldsymbol{\tau}\cdot\mathbf{n},u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

For this problem, we define the operator M as

$$\begin{aligned}
\left\langle M\left(\boldsymbol{\sigma},u\right),\left(\boldsymbol{\tau},v\right)\right\rangle_{W'(\Omega)\times W(\Omega)} &= \left\langle \boldsymbol{\tau}\cdot\mathbf{n},u\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)}\\
&\quad - \left\langle \boldsymbol{\sigma}\cdot\mathbf{n},v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} + 2\int_{\partial\Omega}\lambda uv\,ds.
\end{aligned}$$

Then, it can be shown M satisfies (2.5a) and (2.5b). Furthermore,

$$\begin{aligned}
V &= \left\{(\boldsymbol{\sigma},u)\in H\left(\mathrm{div},\Omega\right)\times H^1\left(\Omega\right) : \boldsymbol{\sigma}\cdot\mathbf{n} = \lambda u \text{ on } \partial\Omega\right\},\\
V^* &= \left\{(\boldsymbol{\sigma},u)\in H\left(\mathrm{div},\Omega\right)\times H^1\left(\Omega\right) : \boldsymbol{\sigma}\cdot\mathbf{n} = -\lambda u \text{ on } \partial\Omega\right\}.
\end{aligned}$$

For any $q = (q^{\boldsymbol{\sigma}}, q^u) \in W(\Omega)$, the above results suggest a natural continuous extension of $(M-B)\,q$ from $W'(\Omega)$ to $W'(\Omega_h)$ as

$$\left\langle (M-B)\,q,(\boldsymbol{\tau},v)\right\rangle_{W'(\Omega_h)\times W(\Omega_h)} = -2\left\langle q^{\boldsymbol{\sigma}}\cdot\mathbf{n},v\right\rangle_{H^{-\frac{1}{2}}(\partial\Omega_h)\times H^{\frac{1}{2}}(\partial\Omega_h)} + 2\int_{\partial\Omega_h}\lambda q^u v\,ds,$$

from which the compatibility condition (2.13) is trivially satisfied.

The study of the quotient space $\tilde{W}(\Omega) = (H(\mathrm{div},\Omega)\times H^1(\Omega))/Q(\Omega)$ is similar to the convection-diffusion-reaction problem in section 3.2, and hence omitted. The abstract DPG formulation (2.14) now equivalently becomes, $\forall\,(\boldsymbol{\tau},v)\in H\left(\mathrm{div},\Omega_h\right)\times H^1\left(\Omega_h\right)$,

$$\begin{aligned}
&\text{Given } (f,g)\in\left[H\left(\mathrm{div},\Omega_h\right)\right]'\times\left[H^1\left(\Omega_h\right)\right]'.\\
&\text{Seek } (\boldsymbol{\sigma},u,\hat{\boldsymbol{\sigma}},\hat{u})\in\left[L^2\left(\Omega_h\right)\right]^d\times L^2\left(\Omega_h\right)\times H^{-\frac{1}{2}}\left(\Gamma_h^0\right)\times H^{\frac{1}{2}}\left(\Gamma_h\right) \text{ such that}\\
&\sum_{j=1}^{N^{\mathrm{el}}}\int_{K_j}\boldsymbol{\sigma}\cdot\left(\varepsilon\boldsymbol{\tau}-\nabla v\right) + u\left(-\nabla\cdot\boldsymbol{\tau}+\mu v\right)\,d\mathbf{x}\\
&\qquad + \left\langle [\![\boldsymbol{\tau}]\!],\hat{u}\right\rangle_{H^{-\frac{1}{2}}(\Gamma_h)\times H^{\frac{1}{2}}(\Gamma_h)} + \left\langle \hat{\boldsymbol{\sigma}},[\![v]\!]\right\rangle_{H^{-\frac{1}{2}}(\Gamma_h^0)\times H^{\frac{1}{2}}(\Gamma_h^0)} + \int_{\partial\Omega_h}\lambda\hat{u}v\,ds\\
&\quad = \left\langle g,v\right\rangle_{[H^1(\Omega_h)]'\times[H^1(\Omega_h)]} + \left\langle f,\boldsymbol{\tau}\right\rangle_{[H(\mathrm{div},\Omega_h)]'\times[H(\mathrm{div},\Omega_h)]}.
\end{aligned} \tag{3.8}$$

Consequently, the results in section 2 hold. Our work is one of the first efforts in developing DPG methods for time-dependent PDEs in general, and the first for time-domain acoustic equations in particular. Since the bilinear form is identical for all time steps, so are the optimal test functions, assuming the trial basis functions are not a function of time. In other words, the optimal test functions, once computed for the first time step, can be used for all subsequent time steps. Another direct consequence is that the stiffness matrix remains the same for all time steps, implying that matrix factorization is only done once if a direct solver is used. Hence, the time-domain DPG methods for acoustic equations proposed in this section are slightly more expensive than the existing DPG methods for steady convection-diffusion problems.

3.5. Maxwell's equations in the elliptic regime. We now apply the abstract theory in section 2 to a version of the Maxwell's equation considered in [21, 24]. The governing equations in three dimensional space, i.e., $d = 3$, read

$$\begin{aligned} \mu H + \nabla \times E &= f \quad \text{in } \Omega, \\ \lambda E - \nabla \times H &= g \quad \text{in } \Omega, \\ E \times \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\mu, \lambda \in L^\infty(\Omega)$ are positive and bounded away from zero. Here, E and H are the electric and the magnetic fields, respectively. Clearly, E, H, f, g are vector-valued functions in $\mathbb{R}^3$. One can cast the governing equations into the Friedrichs' framework discussed in section 2.2 as in [21, 24] with the graph space

$$W(\Omega) = H(\text{curl}, \Omega) \times H(\text{curl}, \Omega).$$

We refer the readers to [21, 24] for the expressions of B and M and the proof that

$$V = V^* = \left\{(H, E) \in H(\text{curl}, \Omega) \times H(\text{curl}, \Omega) : (E \times \mathbf{n})|_{\partial\Omega} = 0\right\}.$$

Now, for any $q = (q^H, q^E) \in W(\Omega)$, the natural extension of $(M - B)q$ from $W'(\Omega)$ to $W'(\Omega_h)$ is defined as

$$\langle (M - B)q, (h, e)\rangle_{W'(\Omega_h)\times W(\Omega_h)} = \sum_{j=1}^{N^{\text{el}}} -2\left(\nabla \times q^E, h\right)_{[L(K_j)]^3} + 2\left(q^E, \nabla \times h\right)_{[L(K_j)]^3}.$$

Thus, the compatibility condition (2.13) is automatically satisfied.

Next, we study the quotient space $\widetilde{W}(\Omega) = H(\text{curl}, \Omega) \times H(\text{curl}, \Omega)/Q(\Omega)$. Here is a result parallel to Theorem 3.2.

THEOREM 3.8.

(i) *The subspace Q is given by*

$$Q = \left\{q \in H(curl, \Omega) \times H(curl, \Omega) : q^H \times \mathbf{n} = 0 \text{ on } \Gamma_h\right\}.$$

Furthermore, $(H(curl, \Omega) \times H(curl, \Omega))/Q(\Omega)$ is isomorphic to $H^{-\frac{1}{2}}(\Gamma_h)$. In particular, the trace of a function in the quotient space $(H(curl, \Omega) \times H(curl, \Omega))/Q(\Omega)$ is independent of its representations.

(ii) *For each $\hat{H} \in H^{-\frac{1}{2}}(\Gamma_h)$, define a new norm*

$$\|\hat{H}\|_{H^{-\frac{1}{2}}(\Gamma_h)} = \|[q]\|_{(H(curl,\Omega)\times H(curl,\Omega))/Q(\Omega)},$$

where $[q] \in (H(curl, \Omega) \times H(curl, \Omega))/Q(\Omega)$ such that there exists a repre-

sentation q satisfying $q^H \times \mathbf{n} = \hat{H}$ on Γ_h. Then, $\llbracket \cdot \rrbracket_{H^{-\frac{1}{2}}(\Gamma_h)}$ is equivalent to $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h)}$, and hence generating the same topology in $H^{-\frac{1}{2}}(\Gamma_h)$. In particular, $(H(curl,\Omega) \times H(curl,\Omega))/Q(\Omega)$ and $H^{-\frac{1}{2}}(\Gamma_h)$ are homeomorphic.

Proof. For $(h,e) \in H^1(\Omega_h) \times H^1(\Omega_h) \subset H(\mathrm{curl},\Omega_h) \times H(\mathrm{curl},\Omega_h)$, the bilinear form $a(q,(h,e))$ becomes

$$a(q,(h,e)) = -\sum_{e \in \Gamma_h} \left(q^H \times \mathbf{n}_e, \llbracket e \rrbracket\right)_{H^{-\frac{1}{2}}(e) \times H^{\frac{1}{2}}(e)}.$$

Now enforcing $a(q,(h,e)) = 0$ for all $(h,e) \in H^1(\Omega_h) \times H^1(\Omega_h) \subset H(\mathrm{curl},\Omega_h) \times H(\mathrm{curl},\Omega_h)$ concludes that $q^H \times \mathbf{n} = 0$ on Γ_h since the trace of $H^1(\Omega_h)$ spans $H^{\frac{1}{2}}(\Gamma_h)$. The rest of the proof is similar to that of Theorem 3.2. □

Theorem 3.8 suggests that we can identify $q \in (H(\mathrm{curl},\Omega) \times H(\mathrm{curl},\Omega))/Q(\Omega)$ with $\hat{H} \in H^{-\frac{1}{2}}(\Gamma_h)$, and we can use either $\llbracket \cdot \rrbracket_{H^{-\frac{1}{2}}(\Gamma_h)}$ or $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_h)}$ as norm in $H^{-\frac{1}{2}}(\Gamma_h)$. Unlike other problems in previous sections, the new unknown q cannot be substituted by $\hat{H}$ since B_{K_j} does not generally have a boundary representation when $(h,e) \in H(\mathrm{curl},\Omega_h) \times H(\mathrm{curl},\Omega_h)$. This is, however, possible if (h,e) is restricted in $H^1(\Omega_h) \times H^1(\Omega_h)$ (see Lemma 2.2 and (2.8)). It should be emphasized here that a boundary representation is vital for finite dimensional approximations since one needs to solve for the unknown flux $\hat{H}$ on the skeleton Γ_h instead of q on the whole domain Ω as we now show. Suppose the subspace $\mathcal{V}_r$ introduced in section 2.4 is a subset of $H^1(\Omega_h)$, then the discrete equation (2.21) equivalently reads

$$\begin{aligned}
&\text{Given } (f,g) \in [H(\mathrm{curl},\Omega_h) \times H(\mathrm{curl},\Omega_h)]'. \\
&\text{Seek } \left(u,\hat{H}\right) \in \mathcal{U}_N \subset L(\Omega_h) \times H^{-\frac{1}{2}}(\Gamma_h) \text{ such that, } \forall (h,e) \in \mathcal{V}_r, \\
&\sum_{j=1}^{N^{\mathrm{el}}} \int_{K_j} \left(-u^E \cdot \nabla \times h + u^H \cdot \nabla \times e\right) \, d\mathbf{x} - \left\langle \hat{H}, \llbracket e \rrbracket \right\rangle_{H^{-\frac{1}{2}}(\Gamma_h) \times H^{\frac{1}{2}}(\Gamma_h)} \\
&\quad = \langle f,h \rangle_{[H(\mathrm{curl},\Omega_h)]' \times H(\mathrm{curl},\Omega_h)} + \langle g,e \rangle_{[H(\mathrm{curl},\Omega_h)]' \times H(\mathrm{curl},\Omega_h)}.
\end{aligned} \tag{3.9}$$

4. Conclusions. We have proposed a unified framework for the discontinuous Petrov–Galerkin method of Demkowicz and Gopalakrishnan [13, 15] based on Friedrichs-like systems, which embrace a large class of elliptic, parabolic, and hyperbolic PDEs. The well-posedness, i.e., existence, uniqueness, and stability, of the DPG solution is established on a single abstract DPG formulation, and two abstract DPG methods corresponding to two different, but equivalent, norms are devised. We have then applied the single DPG framework to several linear(ized) PDEs including, but not limited to, scalar transport, Laplace, diffusion, convection-diffusion, convection-diffusion-reaction, linear(ized) continuum mechanics (e.g., linear(ized) elasticity, a version of the linearized Navier–Stokes equations, etc.), time-domain acoustics, and a version of the Maxwell's equations. The results show that we not only recover several existing DPG methods, but also discover new DPG methods for both PDEs currently considered in the DPG community and new ones. As a direct consequence of the single abstract DPG framework, all of the resulting DPG methods have been shown to be trivially well-posed.

Ongoing research is to apply the abstract framework to the linearized Euler and compressible Navier–Stokes equations. On the other hand, since the setting is in real Hilbert spaces, our methodology cannot be directly applied to the Helmholtz equations. One of our future directions is therefore to modify the theory to complex Hilbert spaces.

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