Equivalence Theorems and Their Applications

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Time Domain Electromagnetic Waves

Spherical Cavity

electromagnetic scattering
Consistency, Stability and Convergence
   An Equivalence Theorem for $y = T x$: $T$ linear
   Lax Equivalence Theorem for $u'(t) = A u(t)$
   Nonlinear maps $y = T x$
   Equivalence Theorem for $A y = x$

A Discontinuous Spectral Element Method for Hyperbolic Equations?? (Next presentation)

Stability of a Discontinuous Spectral Element Method for Wave Propagation Problems (Next presentation)
Consistency, Stability and Convergence

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Motivation

Question
What is the first thing you need to do when you derive/invent a new numerical method?
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What is the first thing you need to do when you derive/invent a new numerical method?

- consistency
- stability
- convergence
Consistency, Stability and Convergence

What is consistency?

What is stability?

What is convergence?
Consistency, Stability and Convergence

What is consistency?

- **Consistency** is a measure of how close a discretization is to the “continuous” problem = how good you approximate operators and functions

What is stability?

- Stability means that the propagated error is controlled by the error in the data = continuity of solution w.r.t the data, uniform boundedness of the discrete operator

What is convergence?

- Convergence means that the discrete solution converges to the exact solution = error between the exact and discrete solutions converges to zero
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The importance of equivalence theorems

Which of the three (consistency, stability, convergence) is the most difficult? Why?
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P. Lax 1953

"Well-posedness of the original differential equation problem and consistency imply the equivalence between stability and convergence of difference methods"
The importance of equivalence theorems

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- Convergence: needs knowledge about the exact solution

Alternate route for convergence: Equivalence theorems

\[ \text{consistency} + \text{stability} \rightarrow \text{convergence} \]
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Which of the three (consistency, stability, convergence) is the most difficult? Why?

- Convergence: needs knowledge about the exact solution

Alternate route for convergence: **Equivalence theorems**

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\text{consistency} + \text{stability} \rightarrow \text{convergence}
\]

Peter Lax 1953

“*Well-posedness of the original differential equation problem and consistency imply the equivalence between stability and convergence of difference methods*”

\[
\text{stability} \iff \text{convergence}
\]
More on Stability

The “easiest” among the three?
More on Stability

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- purely the property of the discrete problem: Knowledge about the exact solution/operators is not needed
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- An analogy: uniqueness implies existence (matrix theory, Fredholm theory). Think about the proof of Banach fixed point theorem. (next talk about inverse problem theory)
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► An analogy: *uniqueness implies existence* (matrix theory, Fredholm theory). Think about the proof of Banach fixed point theorem. *(next talk about inverse problem theory)*

Extremely important for computer implementation?
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Extremely important for computer implementation?

- round-off errors
Definitions
Let $V, W$ be Banach spaces, and $T, T_h : V \to W$

i) Wellposedness: continuity of $T, T_h$

ii) Consistency: $T_h$ is said to be consistent with $T$ if
\[ \lim_{h \to 0} \| (T_h - T) v_0 \| = 0, \forall v_0 \in D \subset V, D \text{ dense in } V, \]

iii) Stability: $T_h$ is called stable if $\sup_h \| T_h \| < \infty$. (Uniform boundedness)

iv) Convergence: $T_h$ is said to converge to $T$, if
\[ \lim_{h \to 0} \| (T_h - T) v \| = 0, \forall v \in V \]

Replace $h$ by $n$ if $n$ is more natural
Equivalence Theorem for Linear Operators

Theorem

A consistent family of $T_h$ is convergent if and only if it is stable.
Equivalence Theorem for Linear Operators

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A consistent family of $T_h$ is convergent if and only if it is stable.

Proof.

$\Rightarrow$) Since $\lim_{h \to 0} \|(T_h - T)v\| = 0$, $\forall v \in V$, the family $T_h$ is pointwise uniformly bounded continuous linear operators. The uniform boundedness principle yields $\sup_h \|T_h\| < \infty$, which is exactly stability.
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$\Leftarrow$ By triangle inequality (three-$\epsilon$ argument), $v_0 \in D$,

$$\|Tv - T_hv\| \leq \|T\| \|v - v_0\| + \|(T - T_h)v_0\| + \|T_h\| \|v_0 - v\|.$$  

The proof is complete by the following two facts. First, the consistency implies $\exists h_0 : \forall h \leq h_0$ such that $\|(T - T_h)v_0\| \leq \epsilon/3$. Second, the density of $D$ allows us to pick $v_0$ such that $\|v - v_0\| \leq \frac{\epsilon}{3 \max\{\|T\|, \sup_h \|T_h\|\}}$. 

$\square$
Numerical Integrations

A suitable setting

- \( V = (C[a, b], \| \cdot \|_\infty) \)

- \( Tf = \int_a^b f \, dx, \quad f \in V \), then \( T \) is linear and bounded, \( |Tf| \leq (b - a) \| f \|_\infty \)

- \( T_n f = \sum_{i=1}^{n} w_i f_i, \quad f \in V \), then \( T_n \) is linear and bounded, \( |T_n f| \leq (\sum_{i=1}^{n} w_i) \| f \|_\infty \)

- The family \( T_n \) is consistent if \( \forall f \in \mathcal{P}, \mathcal{P} \) space of polynomials \((\mathcal{P} \subset V, \text{dense ?})\), \( \lim_{n \to \infty} |T_n f - Tf| = 0 \).

- The family \( T_n \) is stable if \( \sup_n \| T_n \| < \infty \).

- The family \( T_n \) is convergent if \( \forall f \in V \)
  \( \lim_{n \to \infty} |T_n f - Tf| = 0 \)

Examples

Stable : Trapezoidal, Simpson, Gauss quadrature, and etc

Unstable : Newton-Cotes
Numerical Derivatives

A suitable setting

- \( \mathbf{V} = (C^k[a, b], \| \cdot \|_{C^k}) \), \( \mathbf{W} = (C[a, b], \| \cdot \|_\infty) \)

- \( D^{(k)}f, \quad f \in \mathbf{V} \) is linear and bounded, \( \| D^{(k)}f \|_\infty \leq \| f \|_{C^k} \)

- Denote \( D^{(k)}_hf, \quad f \in \mathbf{V} \) the numerical derivative

- The family \( D^{(k)}_h \) is consistent if
  \[
  \lim_{h \to 0} \| (D^{(k)}_h - D^{(k)}) f \|_\infty = 0 \quad \forall f \in \mathbf{D} \subset \mathbf{V}, \mathbf{D} \text{ is dense}
  \]

- The family \( D^{(k)}_h \) is stable if \( \sup_h \| D^{(k)}_h \|_\infty < \infty \).

- The family \( D^{(k)}_h \) is convergent if \( \forall f \in \mathbf{V} \)
  \[
  \lim_{h \to 0} \| D^{(k)}_h f - D^{(k)} f \|_\infty = 0
  \]

Examples: Stability of forward differentiation

\[
\left\| D^{(1)}_h \right\| = \sup_{\| f \|_{C^1} = 1} \sup_x \left| \frac{f(x+h) - f(x)}{h} \right| = \sup_{\| f \|_{C^1} = 1} \sup_x |f'(x + \theta h)| \leq 1
\]
Numerical Derivatives

Convergence is independent of Stability

\[
\lim_{h \to 0} \left\| D_h^{(1)} f - D^{(1)} f \right\|_\infty = \lim_{h \to 0} \sup_x \left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| = \\
\lim_{h \to 0} \sup_x \left| f'(x + \theta h) - f'(x) \right| = 0
\]
Linear time dependent problems

One step method
Consider the problem $u'(t) = Au(t), \ u(0) = u_0$, and an abstract one step method

$$v(h) = B_h u_0,$$
$$v(nh) = B_{nh} u_0.$$

Well-posedness of the continuous problem
Let $S : V \rightarrow V, u(t) = S(t)u_0$, we require $u(t)$ is continuous w.r.t $t$ and $\sup_t \|S(t)\| < \infty$.

Well-posedness of the discrete problem
For each $0 \leq h \leq h_0$, we require $\sup_h \|B_h\| < \infty$.
Linear time dependent problems

Consistency
∀v₀ ∈ D ⊂ V, D is dense,

\[ \lim_{h \to 0} \| B_h u(t) - S(t + h)v_0 \| = 0 \]

Stability
\[ \| B^n_h \| < \infty, \forall h, n : nh \leq T \]

Convergence
\[ \lim_{k \to \infty} \| B^{n_k}_{h_k} v_0 - S(t)v_0 \| = 0, \text{ where } \lim_{k \to \infty} n_k h_k = t \]
Stability $\iff$ Convergence

Stability $\Rightarrow$ Convergence

Using triangle inequality and three-$\epsilon$ trick we have

$$
\| v(nh) - u(t) \| = \| B^n_h u_0 - S(t)u_0 \| \leq \\
\underbrace{\| B^n_h \| \| v - u_0 \|}_\text{stability + density} + \underbrace{\| B^n_h v - S(t)v \|}_\text{consistency} + \underbrace{\| S(t) \| \| v - u_0 \|}_\text{wellposedness + density} \to 0,
$$
Stability $\iff$ Convergence

**Convergence $\Rightarrow$ Stability**

If $n$ is finite then by the discrete wellposedness we have

$$\sup_h \|B_n^h\| \leq \sup_h \|B_h^n\|^n < \infty.$$

Now $k \to \infty : \lim_k n_k h_k = t$, by convergence we have for each $u_0 \in V$, the family $B_{h_k}^{n_k}$ is uniformly bounded. Then by the uniform boundedness principle, we have

$$\sup_k \left\| B_{h_k}^{n_k} \right\| < \infty.$$

In both cases, the stability condition is proved.
A nonlinear setting for $y = Tx$

$T$ is nonlinear

$T : V \to W$, and $V, W$ are Banach

- **Wellposedness**: $T$ is continuous
- **Consistency**: convergence on a dense subspace $D$

\[ \forall v \in D \subset V : \lim_{n \to \infty} \|T_n v - T v\| = 0 \]

- **Stability** $\forall \epsilon, \|v - v_0\| \leq \delta$, such that $\|T_n v - T_n v_0\| \leq \epsilon$
  (replace uniform boundedness by equi-continuity)

- **Convergence**

\[ \forall v \in V, \exists n_0 : n \geq n_0 \lim_{n \to \infty} \|T_n v - T v\| = 0 \]
Theorem

A consistent family of $T_h$ is convergent if and only if it is stable.
Equivalence Theorem for nonlinear Operators

Theorem

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Proof.

$\Rightarrow$) Again the three-$\epsilon$ trick, $\|v - v_0\| < \delta$

$$\|T_n v - T_n v_0\| \leq \underbrace{\|T_n v - T v\|}_{\text{convergence}} + \underbrace{\|T v - T v_0\|}_{\text{Wellposedness}} + \underbrace{\|T v_0 - T_n v_0\|}_{\text{convergence}} \leq \epsilon$$
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⇐) By the three-$\epsilon$ trick, $\forall v \in V$, by density $\exists v_0 \in D$, $\|v - v_0\| < \delta$

\[
\|T v - T_n v\| \leq \underbrace{\|T v - T v_0\|}_{\text{wellposedness}} + \underbrace{\|T v_0 - T_n v_0\|}_{\text{Consistency}} + \underbrace{\|T_n v_0 - T_n v\|}_{\text{Stability}} \leq \epsilon
\]
Linear equation $Ay = x$

Wellposedness

Let $A : W \rightarrow V$, and $V, W$ are Banach. The problem is wellposed if $A$ is continuous and bijective. We know that the inverse exists, i.e. $T = A^{-1}$, and $T$ is continuous (Why??).

The problem can be converted to the form $y = Tx$ which we have already discussed.
Consistency, Stability and Convergence

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- Nonlinear maps $y = Tx$
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