Surface Tension Driven Thin Film Flow

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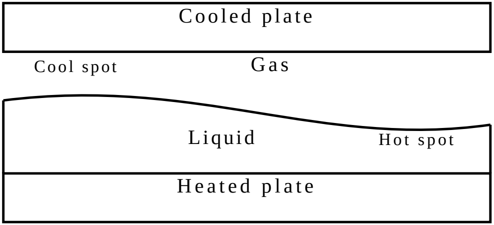
Thin Film Flow



Two Layer Film

- Liquid heated from flat solid surface below
- Gas cooled from flat plate above
- Experiments + Theory from VanHook, Schatz, Swift, McCormick, Swinney
- Theory + Calculations from Wang, Carey, Stogner

Flow Instabilities

Buoyancy cells	Raleigh Number $\mathrm{Ra} \equiv \frac{\alpha_T g \delta T d^3}{\nu k}$ Buoyancy / viscosity	d^3
Thermocapillary cells	Marangoni Number $\mathrm{M} \equiv \frac{\sigma_T \delta T d}{\rho \nu k}$ Surface tension / viscosity	d
Thermocapillary deformation	Inverse dynamic Bond Number $D \equiv \frac{\sigma_T \delta T}{\rho g d^2}$ Surface tension / gravity	d^{-2}

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Two-Dimensional Flow

- $d/L << 1$
- Flow is described by velocity $\vec{v}(x, y)$, pressure $P(x, y)$, temperature T
- Viscosity gives velocity profiles in z defined by $\vec{v}_{max}(x, y)$ or $\overline{v}(x, y)$
- Free surface flows determine pressure from surface height $u(x, y)$ + conditions
- Long wavelength thermocapillary flow describes $\vec{v}$, T in terms of u

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Non-dimensionalization

Based on:

- representative thickness d
- density ρ
- thermal diffusivity $\kappa \equiv \frac{k}{\rho c_p}$
- viscosity ν

Length	$u, \vec{x}$	d
Time	t	$\frac{d^2}{\kappa}$
Velocity	$\vec{v}$	$\frac{\kappa}{d}$
Pressure	P	$\frac{\rho\nu\kappa}{d^2}$
Temperature	T	δT
Surface Tension	S	$\frac{d}{\rho\nu k}$

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Non-dimensionalized Incompressible Flow

$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} &= \mathrm{Pr}\left(-\nabla P + \Delta \vec{v} - \mathrm{G}\hat{e}_z\right) \\ \nabla \cdot \vec{v} &= 0 \\ \frac{\partial T}{\partial t} + (\vec{v} \cdot \nabla)\, T &= \Delta T \end{aligned}$$

with Prandtl number $\mathrm{Pr} \equiv \nu/\kappa$ and Galileo number $\mathrm{G} \equiv \frac{gd^3}{\nu\kappa}$

Temperature Boundary Conditions

Lower: Fixed by hot plate

$$T(z=0) \quad = \quad 0$$

Upper: $T(z=u)$ determined by conductivities in z

$$
\begin{aligned}
k\frac{T\delta T}{ud} &= k_g\frac{(1-T)\delta T}{d+d_g-ud} \\
H &\equiv \frac{k_g d}{k d_g} \\
F &\equiv \frac{d/d_g - H}{1+H} \\
T &= \frac{-u}{1+F-Fu}
\end{aligned}
$$

Velocity Boundary Conditions

Lower: No-slip

$$\vec{v}(z=0) = \vec{0}$$

Upper: $P(z=u)$ and $\vec{v}(z=u)$ interact with surface tension

$$\begin{aligned} \nabla_{\perp} S &= \nabla_{\perp} \vec{v}_n + \nabla_n \vec{v}_{\perp} \\ P - S\left(\frac{1}{R_1} + \frac{1}{R_2}\right) &= 2\frac{\partial \vec{v}_n}{\partial x_n} \end{aligned}$$

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Long Wavelength Interior Equations

Expansion in d/L, low order terms dropped.

$$
\begin{aligned}
\frac{\partial^2 \vec{v}_\perp}{\partial z^2} &= \nabla_\perp P \\
\frac{\partial P}{\partial z} &= -\mathrm{G} \\
\frac{\partial^2 T}{\partial z^2} &= 0 \\
\nabla_\perp \cdot \vec{v}_\perp + \frac{\partial \vec{v}_z}{\partial z} &= 0
\end{aligned}
$$

Long Wavelength Surface Equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \vec{v}_\perp \cdot \nabla u &= \vec{v}_z \\
\nabla u \cdot \nabla S - \nabla u \cdot \frac{\partial \vec{v}_\perp}{\partial z} &= 0 \\
\left(\nabla u \times \nabla S - \nabla u \times \frac{\partial \vec{v}_\perp}{\partial z} \right) \cdot \hat{e}_z &= 0 \\
P + S \Delta u &= 0
\end{aligned}
$$

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Integrating Out Pressure

Surface tension + gravity determine pressure:

$$P = -S\Delta u + \mathrm{G}(u - z)$$

Which forces horizontal momentum:

$$\frac{\partial^2 \vec{v}_\perp}{\partial z^2} = -\nabla S\Delta u + \mathrm{G}\nabla u$$

Integrating Out Velocity

Integrating momentum twice in z between boundary conditions:

$$\vec{v}_\perp = \left(-\nabla\left(S\Delta u\right) + \mathrm{G}\nabla u\right)\frac{z^2}{2} + \left(u\nabla S\Delta u - u\mathrm{G}\nabla u + \nabla S\right)z$$

And once more for $\partial_z \vec{v}_z = \nabla_\perp \cdot \vec{v}_\perp$:

$$\vec{v}_z = \left(-\Delta\left(S\Delta u\right) + \mathrm{G}\Delta u\right)\frac{z^3}{6} + \left(\nabla\cdot\left(u\nabla S\Delta u\right) - \mathrm{G}\nabla\cdot\left(u\nabla u\right) + \Delta S\right)$$

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Thin Film Flow Equation

$\vec{v}_z$ and $\vec{v}_\perp$ fully define $\frac{\partial u}{\partial t}$ in terms of u:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \vec{v}_z(u) - \vec{v}_\perp(u) \cdot \nabla u \\
&= \vec{v}_z(u) - \nabla \cdot (u\vec{v}_\perp(u)) + \nabla \cdot \vec{v}_\perp(u) \\
&= \vec{v}_z(u) - \nabla \cdot (u\vec{v}_\perp(u)) - \left.\frac{\partial \vec{v}_z}{\partial z}\right|_{z=h} \\
&= \nabla \cdot \left(\frac{u^3}{3} \nabla (S \Delta u) - \frac{u^2}{2} \nabla S + \frac{\mathrm{G}u^3}{3} \nabla u \right)
\end{aligned}$$

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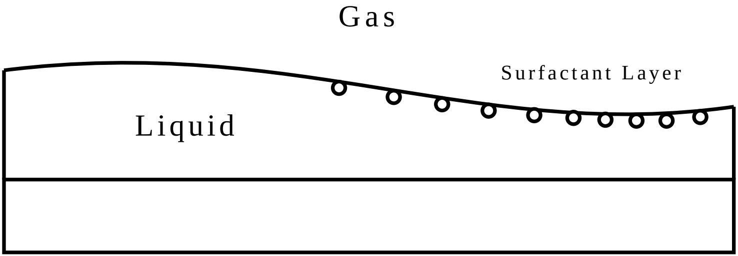
Surfactant Transport



Flow Characteristics

- Temperature, surfactant concentration c_s determine surface tension
- Temperature destabilizes, surfactant stabilizes

Surface Transport Equation

$$\frac{\partial c_s}{\partial t} + \nabla \cdot (c_s \vec{v}_\perp) = 0$$

Expands based on $\vec{v}_\perp$ to:

$$\frac{\partial c_s}{\partial t} + \nabla \cdot \left(c_s \frac{u^2}{2} \left(-\nabla \left(S \Delta u \right) + \mathrm{G} \nabla u \right) + \left(u^2 \nabla S \Delta u - u^2 \mathrm{G} \nabla u + u \nabla S \right) \right) = 0$$

Surfactant-dependent Surface Tension

Dilute surfactant model: linear dependencies on temperature, concentration

$$
\begin{aligned}
S &= S_0 - \mathrm{M}(T - T_0) - \alpha E S_0 c_s \\
S &= S_0 + \frac{\mathrm{M}u}{1 + F - Fu} - \alpha E S_0 c_s \\
\nabla S &= \frac{\mathrm{M}(1+F)}{(1+F-Fu)^2}\nabla u - \alpha E S_0 \nabla c_s
\end{aligned}
$$

Thin Film Flow and Transport Equations

Substituting in constitutive equations for S, ∇S completes the equation system:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{u^3}{3} \nabla \left(\left(S_0 + \frac{\mathrm{M}u}{1+F-Fu} - \alpha E S_0 c_s \right) \Delta u \right) - \frac{u^2}{2} \frac{\mathrm{M}(1+F)}{(1+F-Fu)^2} \nabla u + \frac{u^2}{2} \nabla c_s + \frac{\mathrm{G}u^3}{3} \nabla u \right)$$

$$\frac{\partial c_s}{\partial t} = \nabla \cdot \left(\left(\frac{u^2}{2} \nabla \left(\left(S_0 + \frac{\mathrm{M}u}{1+F-Fu} - \alpha E S_0 c_s \right) \Delta u \right) + \frac{\mathrm{M}(1+F)u}{(1+F-Fu)^2} \nabla u - \alpha E S_0 u \nabla c_s \right) c_s \right)$$

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Thin Film Flow and Transport Equations

Use static Bond number $B \equiv \frac{\rho g L^2}{S_{eq}}$, inverse dynamic Bond number $D \equiv \frac{M}{G}$, Wang number $D_s \equiv \frac{\alpha E S_0}{\rho g d^2}$, rescaling $\vec{x}$ by L/d, t by $\frac{3L^2}{GD^2}$:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla \cdot \left(\frac{u^3}{B} \nabla \left(\left(1 + \frac{Dud^2}{(1+F-Fu)L^2} - D_s \frac{d^2}{L^2} c_s \right) \Delta u \right) - \right. \\
&\quad \left. \frac{3u^2}{2} \frac{D(1+F)}{(1+F-Fu)^2} \nabla u + \frac{u^2}{2} \nabla c_s + \frac{\mathrm{G}u^3}{3} \nabla u \right) \\
\frac{\partial c_s}{\partial t} &= \nabla \cdot \left(\left(\frac{3u^2}{2B} \nabla \left(\left(1 + \frac{Dud^2}{(1+F-Fu)L^2} - D_s \frac{d^2}{L^2} c_s \right) \Delta u \right) + \right. \right. \\
&\quad \left. \left. \frac{D(1+F)u}{(1+F-Fu)^2} \nabla u - D_s u \nabla c_s \right) c_s \right)
\end{aligned}
$$

Thin Film Flow and Transport Equations

Finally, dropping terms scaling with d^2/L^2 gives standard equations:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla \cdot \left(\left(u^3 - \frac{3D(1+F)u^2}{2(1+F-Fu)^2} \right) \nabla u - \right. \\ &\left. \frac{u^3}{B} \nabla \Delta u + \frac{3D_s u^2}{2} \nabla c_s \right) \\ \frac{\partial c_s}{\partial t} &= \nabla \cdot \left(\left(\frac{u^2}{2} - \frac{3D(1+F)u}{(1+F-Fu)^2} \right) c_s \nabla u - \right. \\ &\left. \frac{c_s u^2}{2B} \nabla \Delta u + 3D_s c_s u \nabla c_s \right)\end{aligned}$$

Weak Approximation

Taking weighted residuals of $\frac{\partial u}{\partial t}$, integrating each 2nd order term by parts once and each 4th order term twice,

$$\begin{aligned}
\left(\frac{\partial u}{\partial t}, v\right) \quad = \quad & \left(\left(\frac{3D(1+F)u^2}{2(1+F-Fu)^2} - u^3\right)\nabla u - \frac{3D_s u^2}{2}\nabla c_s, \nabla v\right)_\Omega + \\
& \left(\left(u^3 - \frac{3D(1+F)u^2}{2(1+F-Fu)^2}\right)\partial_{\vec{n}} u + \frac{3D_s u^2}{2}\partial_{\vec{n}} c_s, v\right)_{\partial\Omega} - \\
& \left(\frac{u^3}{B}\Delta u, \Delta v\right)_\Omega - \left(\frac{3u^2}{B}\Delta u \nabla u, \nabla v\right)_\Omega + \\
& \left(\frac{u^3}{B}\partial_{\vec{n}}\Delta u, v\right)_{\partial\Omega} - \left(\frac{u^3}{B}\Delta u, \partial_{\vec{n}} v\right)_{\partial\Omega}
\end{aligned}$$

Weak Approximation

Taking weighted residuals of $\frac{\partial c_s}{\partial t}$, integrating each 2nd order term by parts once and each 4th order term twice,

$$
\begin{aligned}
\left(\frac{\partial c_s}{\partial t}, w\right) = \; & \left(\left(\frac{3D(1+F)u}{(1+F-Fu)^2} - \frac{u^2}{2}\right) c_s \nabla u - 3D_s c_s u \nabla c_s, \nabla w\right)_{\Omega} + \\
& \left(\left(\frac{u^2}{2} - \frac{3D(1+F)u}{(1+F-Fu)^2}\right) c_s \partial_{\vec{n}} u + 3D_s c_s u \partial_{\vec{n}} c_s, w\right)_{\partial\Omega} - \\
& \left(\frac{c_s u^2}{2B} \Delta u, \Delta w\right)_{\Omega} - \left(\Delta u \frac{2 c_s u \nabla u + u^2 \nabla c_s}{2B}, \nabla w\right)_{\Omega} + \\
& \left(\frac{c_s u^2}{2B} \partial_{\vec{n}} \Delta u, w\right)_{\partial\Omega} - \left(\frac{c_s u^2}{2B} \Delta u, \partial_{\vec{n}} w\right)_{\partial\Omega}
\end{aligned}
$$

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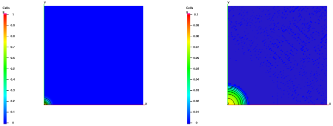
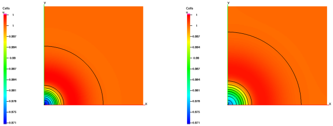
Surfactant-Driven Flow Computation

- Initially flat film surface
- Symmetry BCs
- Surfactant droplet in corner
- Local surface tension reduction
- Fluid wave pushed through domain



Concentration at $t = 0, 0.2$



Depth at $t = 0.1, 0.2$

Stability: Analytic Linearization

Perturbing around steady state $u = 1$ and $c_s = 1$:

$$
\begin{aligned}
u(x, y, t) &= 1 + \delta u = 1 + \Sigma_p \eta_p e^{\gamma_p t} e^{i2\pi \boldsymbol{p}\cdot\boldsymbol{x}} \\
c_s(x, y, t) &= 1 + \delta c_s = 1 + \Sigma_q \xi_q e^{\lambda_q t} e^{i2\pi \boldsymbol{q}\cdot\boldsymbol{x}}
\end{aligned}
$$

Stability: Analytic Linearization

For small δu and δc_s, linearizing gives:

$$\begin{aligned}
\frac{\partial \delta u}{\partial t} &= \nabla \cdot \left(\left(1 - \frac{3D(1+F)}{2} \right) \nabla \delta u - \right. \\
&\qquad \left. \frac{1}{B} \nabla \Delta \delta u + \frac{3D_s}{2} \nabla \delta c_s \right) \\
\frac{\partial \delta c_s}{\partial t} &= \nabla \cdot \left(\left(\frac{1}{2} - 3D(1+F) \right) \nabla \delta u - \right. \\
&\qquad \left. \frac{1}{2B} \nabla \Delta \delta u + 3D_s \nabla \delta c_s \right)
\end{aligned}$$

Stability: Analytic Linearization

Substituting in Fourier expansions allows analytic evaluation of derivatives, leads to algebraic equations for each mode:

$$\gamma_q = \lambda_q$$

$$\gamma_q + 6\pi^2 q^2 D_s \left(\frac{\xi_q}{\eta_q}\right) + 4\pi^2 q^2 \left(-\frac{3}{2}D(1+F) + 1 + \frac{4\pi^2 q^2}{B}\right) = 0$$

$$\lambda_q + 6\pi^2 q^2 \left(-2D(1+F) + 1 + \frac{4\pi^2 q^2}{B}\right) \left(\frac{\xi_q}{\eta_q}\right)^{-1} + 12\pi^2 q^2 D_s = 0$$

for all $\boldsymbol{q} = (1,0), (0,1), (1,1), \ldots$.

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Stability: Analytic Linearization

Solving for time growth rates γ_q and λ_q gives:

$$(\gamma_q)_{1,2} = (\lambda_q)_{1,2} = 2\pi^2 q^2 \left(\varepsilon_q - 3D_s \pm \sqrt{\Delta_q}\right)$$

Based on terms:

$$\Delta_q \equiv (\varepsilon_q - 3D_s)^2 - 3D_s \left(1 + \frac{4\pi^2 q^2}{B}\right)$$

$$\varepsilon_q \equiv \frac{3D(1+F)}{2} - 1 - \frac{4\pi^2 q^2}{B}$$

$\varepsilon_q > 3D_s$: unstable time growth
$\Delta_q < 0$: surfactant-driven oscillation

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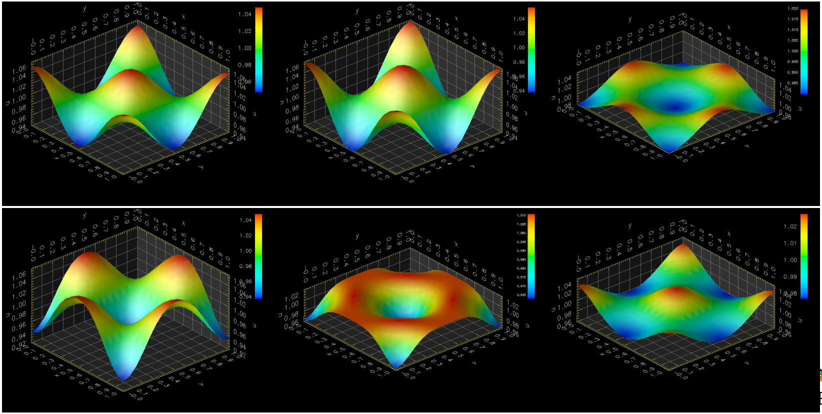
Stability: Computation

$u(t),\, t = 0 \rightarrow 0.4$



Open Questions

- Proper 2D boundary conditions
- FEM Eigenproblem convergence
- Surfactant diffusion effects
- Nonlinear surfactant effects