Adaptive C1 Macroelements for Fourth Order and Divergence-Free Problems

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Incompressible Navier-Stokes

For incompressible, constant density fluid flow:

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} + \mathrm{Re}\,(\vec{u} \cdot \nabla)\,\vec{u} &= -\nabla P + \nabla \cdot \left(\nu \left(\nabla \vec{u} + (\nabla \vec{u})^T\right)\right) \\ \nabla \cdot \vec{u} &= 0 \end{aligned}$$

Enforcing the second equation weakly gives a saddle point problem, which on a stable space gives a an approximately divergence-free solution.

Using the right finite elements, can we enforce the divergence-free condition exactly?

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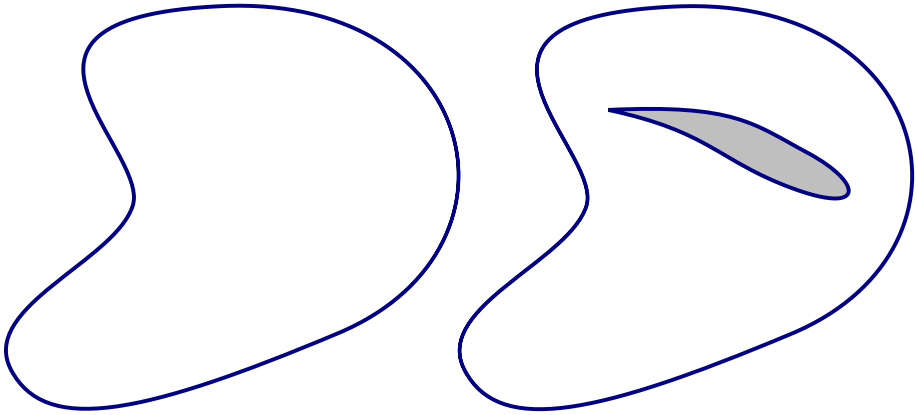
Differential Exact Sequence

$$H^3(\Omega;\mathbb{R}) \xrightarrow{\nabla} H^2(\Omega;\mathbb{R}^d) \xrightarrow{\nabla\times} H^1(\Omega;\mathbb{R}^d) \xrightarrow{\nabla\cdot} H^0(\Omega;\mathbb{R})$$

Every exact differential form is closed; is every closed form exact?

Yes, if Ω is contractable or if every admissible closed form is extensible to a closed form on a contractable $\tilde{\Omega}$.

In such domains, we can generate a basis for divergence-free functions by taking the curl of a basis for arbitrary (sufficiently smooth) functions: $\vec{u} \equiv \nabla \times \vec{\psi}$



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Divergence-Free Formulation

In a divergence-free space, the pressure variable disappears from the weak momentum equation entirely:

$$\int_\Omega \nabla P \cdot \vec{v}\, d\Omega \;=\; \int_{\partial\Omega} P\vec{v}\cdot\vec{n}\, dS - \int_\Omega P\nabla\cdot\vec{v}\, d\Omega = 0$$

For linear Stokes flow, this is equivalent to the streamfunction formulation.

For non-Newtonian fluids, divergence-free elements produce a simpler weak form of the momentum equation with variable viscosity ν:

$$\begin{aligned}\int_\Omega \frac{\partial\vec{u}}{\partial t}\vec{v}\, d\Omega \;=\;& -\mathsf{Re}\int_\Omega \vec{v}(\vec{u}\cdot\nabla)\vec{u}\, d\Omega - \int_\Omega \nu(\nabla\vec{u} + (\nabla\vec{u})^T) : \nabla\vec{v}\, d\Omega \\ & + \int_{\partial\Omega} \nu(\nabla\vec{u} + (\nabla\vec{u})^T)\vec{v}\cdot\vec{n}\, dS\end{aligned}$$

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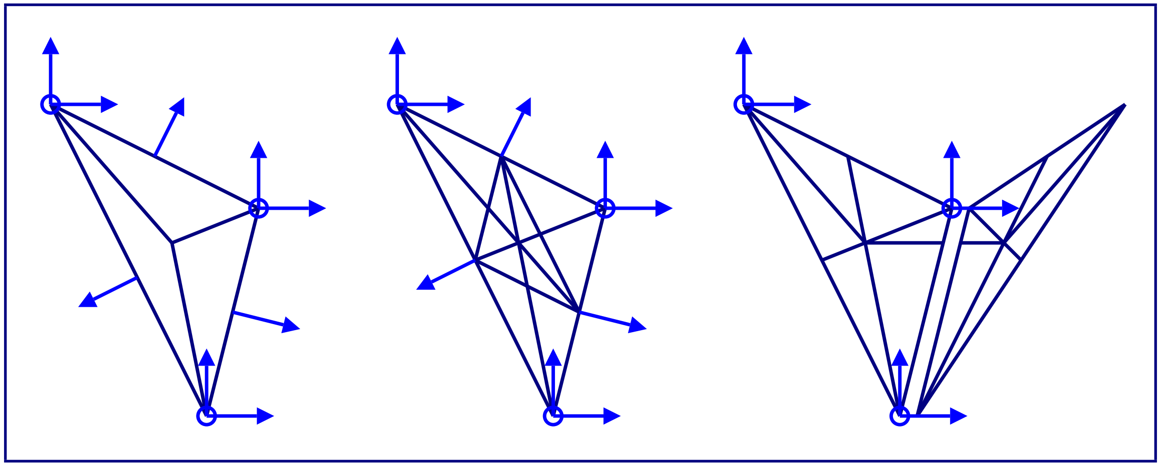
Macroelements

Constraining a polynomial triangle to C^1 continuity along edges requires a high polynomial degree (quintics). Instead, we can construct a macroelement by subdividing the triangle, using piecewise polynomial functions, and adding continuity constraints on interior edges.



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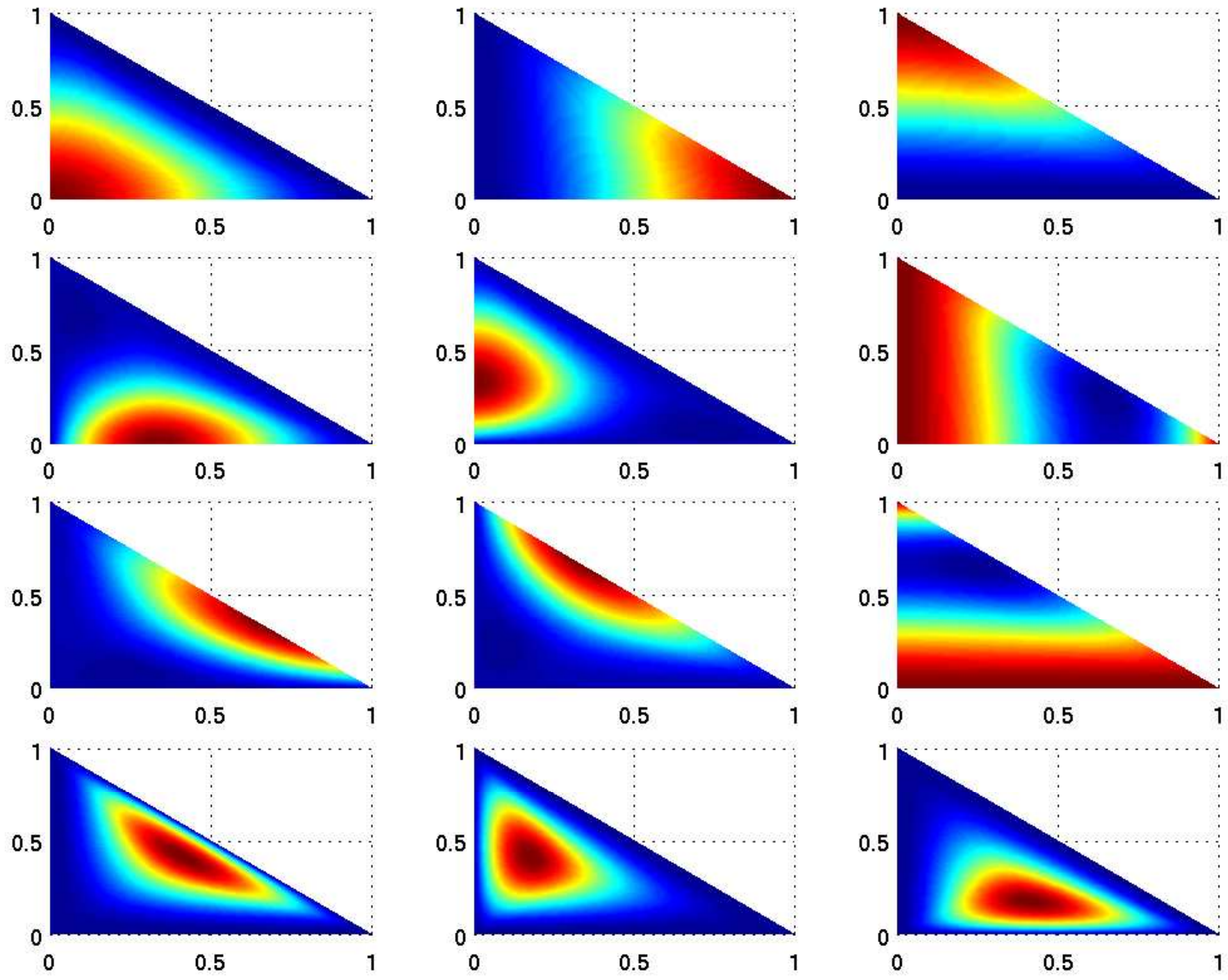
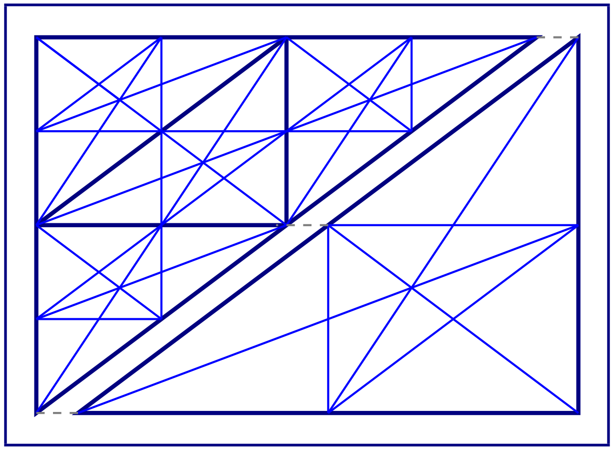
Clough-Tocher 3-split



Hanging Node Compatibility

Adaptive refinement of macroelements requires compatibility between element refinement and element subdivision schemes.



Some macroelements (e.g. Clough-Tocher and Powell-Sabin-Heindl triangles, Alfeld tetrahedra) can be made compatible with non-conforming meshes; others (e.g. Worsey-Farin and Worsey-Piper tetrahedra) cannot.

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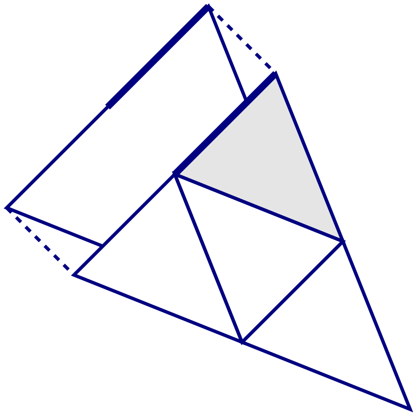
Hanging Node Constraints

Adaptive mesh refinement creates a non-conforming mesh. Forming a C^1 conforming function space requires constraining "hanging node" degrees of freedom on fine elements in terms of degrees of freedom on coarse parent elements.



$$\begin{aligned} u^F\big|_\gamma &= u^C\big|_\gamma \\ \sum_i u_i^F \phi_i^F\Big|_\gamma &= \sum_j u_j^C \phi_j^C\Big|_\gamma \\ A_{ki}u_i &= B_{kj}u_j \\ u_i &= A_{ki}^{-1}B_{kj}u_j \end{aligned}$$

These matrices can be defined using an appropriate inner product:

$$\begin{aligned} A_{ki} &\equiv (\phi_i^F, \phi_k^F)_\gamma \\ B_{kj} &\equiv (\phi_j^C, \phi_k^C)_\gamma \end{aligned}$$

(L_2 integrated values for C^0 continuity, values and fluxes for C^1 continuity)

Coarsening Projections

Adaptivity in time dependent codes requires the ability to refine and coarsen elements while preserving functions in the finite element space.

For projection to a nested refined element, local L_2 projection gives an exact (within FP error) result.

In general, a uniquely defined but locally computable result takes several steps:

- Equate nodal degrees of freedom
- Project edge degrees of freedom, holding nodal values fixed
- Project face degrees of freedom, holding edge values fixed
- Project volume degrees of freedom, holding face values fixed

For C^1 elements, we project values and gradients.

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Error Estimation

Kelly flux jump obviously fails on C^1 elements, but biharmonic problems have analogous error terms:

$$\eta_K \equiv \sum_{S \subset K} \left[\left|\left| f - \Delta^2 u_h \right|\right|_S h_S^2 + \frac{1}{2} \left|\left| [\![\partial_{\vec{n}} \Delta u_h]\!] \right|\right|_{\partial S} h_S^{3/2} + \frac{1}{2} \left|\left| [\![\Delta u_h]\!] \right|\right|_{\partial S} h_S^{1/2} \right]$$

We still have a useful indicator even after dropping higher order terms and integrating only along macroelement boundaries:

$$\tilde{\eta_K} \equiv \left[\frac{1}{2} \left|\left| [\![\Delta u_h]\!] \right|\right|_{\partial K} h_S^{1/2} \right]$$

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Shear-dependent Viscosity

Many fluid constitutive equations require shear-dependent, non-Newtonian fluid models:

Where viscosity ν is solely a function of the shear rate $s \equiv 2\sqrt{D(\vec{u}) : D(\vec{u})}$, it is convenient to reformulate the entire viscosity term in terms of this symmetric rate of deformation tensor $D(\vec{u}) \equiv \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$ rather than the asymmetric velocity gradient $\nabla\vec{u}$:

$$\begin{aligned} \int_\Omega \frac{\partial \vec{u}}{\partial t}\vec{v}\,d\Omega &= -\mathrm{Re}\int_\Omega \vec{v}(\vec{u}\cdot\nabla)\vec{u}\,d\Omega - \int_\Omega 2\nu(s)D(\vec{u}) : D(\vec{v})\,d\Omega \\ &\quad + \int_{\partial\Omega} 2\nu(s)D(\vec{u})\vec{v}\cdot\vec{n}\,dS \end{aligned}$$

The identity $(A + A^T) : B = \frac{1}{2}(A + A^T) : (B + B^T)$ makes this equivalent.

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Successive Approximation

For non-Newtonian fluids, we have two nonlinearities in the Navier-Stokes equations, both viscosity related. The Reynolds number (based on ν_0 at low flow rates) adds a nonlinear convection term for $\mathrm{Re} > 0$, and viscosity variation (based on viscosity ν_∞ at high flow rates and nondimensionalized as ν_∞/ν_0) makes the diffusion term nonlinear for $\nu_\infty \neq \nu_0$.

Rapidly obtaining highly accurate solutions requires a quadratically converging technique like Newton or (with appropriate linear tolerances) Newton-Krylov, but a large region of convergence is obtained by just lagging nonlinear terms:

$$\mathrm{Re} \int_\Omega \vec{v} \cdot (\vec{u}^{(k-1)} \cdot \nabla)\vec{u}^{(k)} \, d\Omega + \int_\Omega 2\nu(s(D(\vec{u}^{(k-1)})))D(\vec{u}^{(k)}) : D(\vec{v}) \, d\Omega \\ - \int_{\partial\Omega} 2\nu(s(D(\vec{u}^{(k-1)})))D(\vec{u}^{(k)})\vec{v} = 0$$

Iteration Convergence

Theoretically, successive approximation of Stokes flow ($\mathrm{Re} = 0$) converges for any shear-thinning fluid with

$$\begin{aligned} \nu_0 &< \infty \\ \nu(s) &\geq \nu_\infty > 0 \end{aligned}$$

Experimentally, successive approximation of Navier-Stokes flow converges given a sufficiently accurate starting iterate (e.g. an approximate solution at a slightly lower Reynolds number), but diverges from inaccurate starting iterates (e.g. zero flow or approximate solutions at much lower Reynolds numbers).

Lower ν_∞/ν_0 ratios shrink the region of convergence, requiring increasingly accurate initial guesses.

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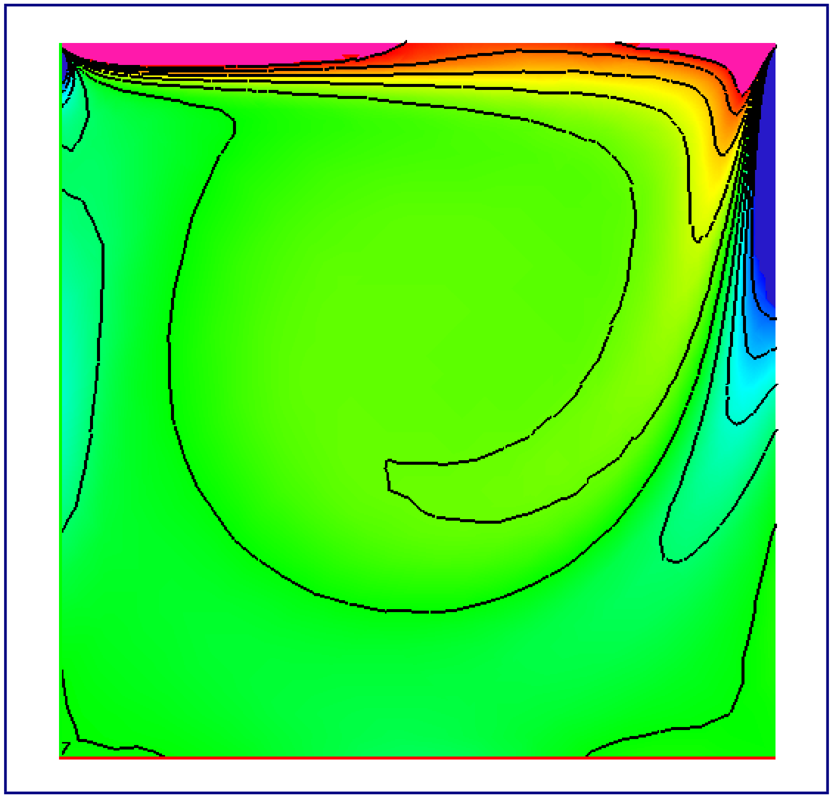
Lid-Driven Cavity

In a more adaptivity-friendly test, we examine a square domain with zero velocity on three sides and a constant lateral velocity along the top.



Corner Singularities

Because the velocity boundary conditions are discontinuous at the corners, the streamfunction's first derivative is also discontinuous and its second derivatives are singular. As $r \rightarrow 0$,

$$\begin{aligned}
\psi(r,\theta) &= \frac{r}{\pi^2-4}\left(-\pi^2\sin(\theta)+2\pi\theta\sin(\theta)+4\theta\cos(\theta)\right) \\
\omega(r,\theta) &= \frac{1}{(\pi^2-4)r}\left(4\pi\cos(\theta)-8\sin(\theta)\right)
\end{aligned}$$

Without removing these singularities analytically, finite element simulations at high Reynolds numbers and on coarse meshes are often corrupted by numerical oscillation.

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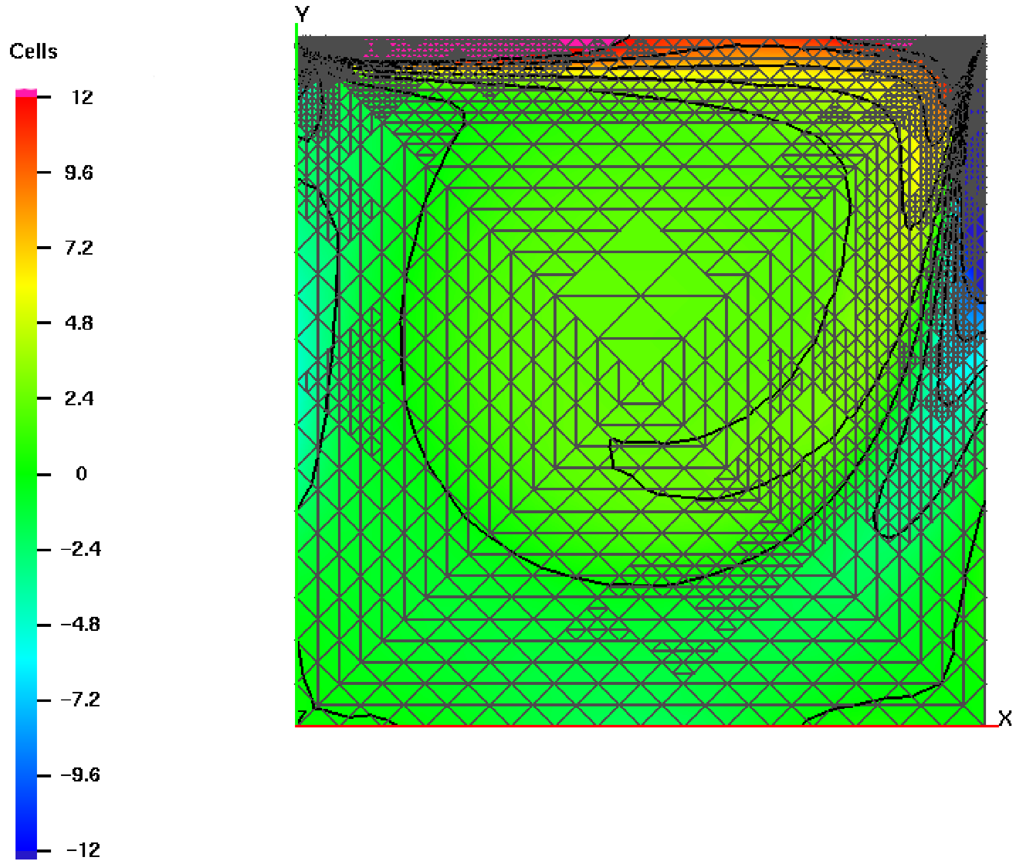
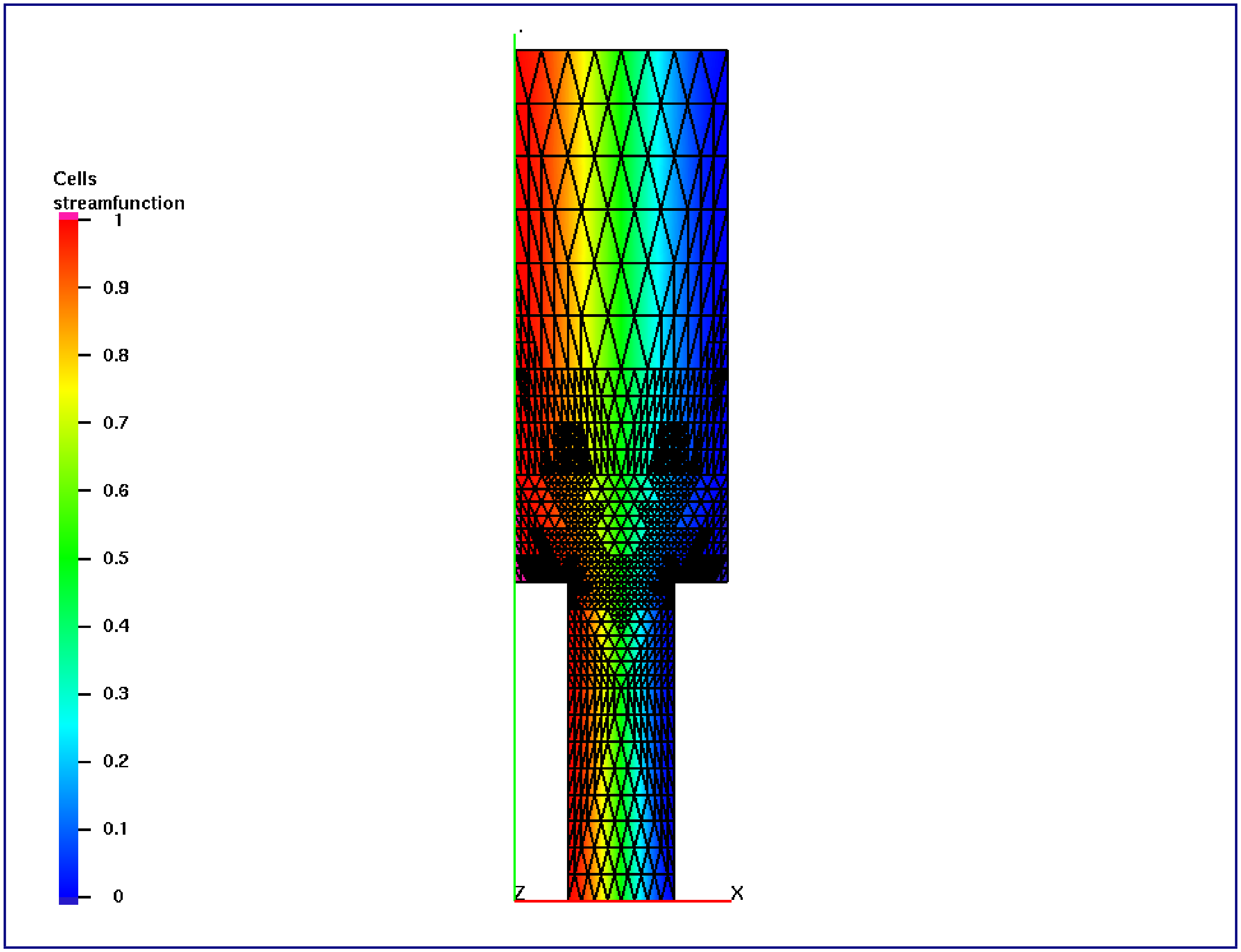
Adaptive Cavity Solution

Re $= 500$, Extended Williamson fluid, $\nu_\infty/\nu_0 = 0.1$



Adaptive Step Solution

$\mathrm{Re} = 0$, Newtonian fluid



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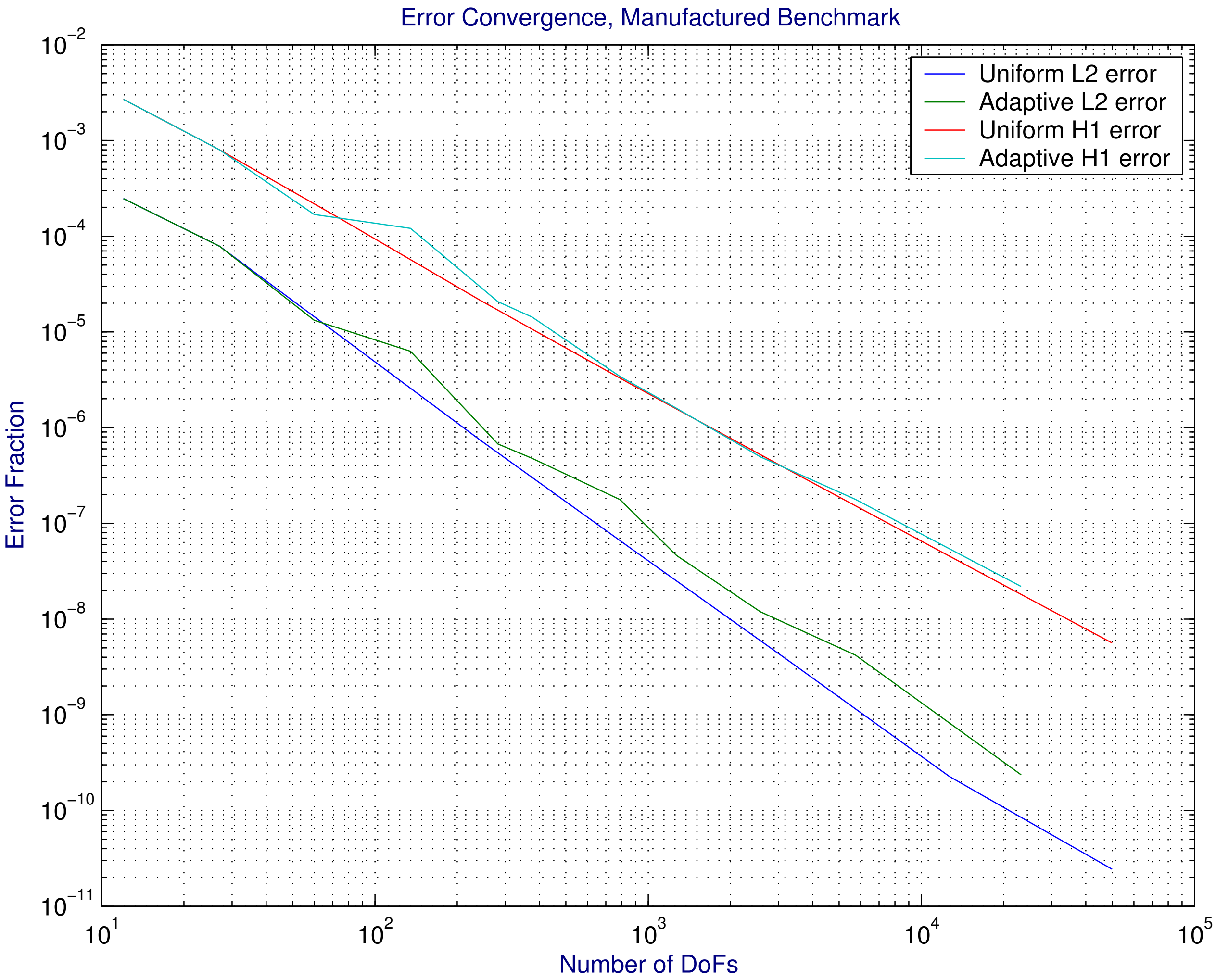
Benchmark Convergence

$$\Delta^2 u = f,\; u = \sin xy$$



Driven Cavity Convergence

Re $= 500$, Extended Williamson fluid, $\nu_\infty/\nu_0 = 0.1$