Math 222a, Second midterm, solutions
Fall 2004

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1 (a) No. $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in H$, but $-e_2 \notin H$.

(b) $Col(A) = \mathbb{R}^4$

(c) $\det(A - \lambda I) = 2\lambda - 4\lambda^2 + 2\lambda^3 = 2\lambda(1 - 2\lambda + \lambda^2) = 2\lambda(1-\lambda)^2$

$\lambda = 0, 1$

(d) $\dim(Nul(A)) = n - k$

(e) $\dim(Nul(A - 3I)) = 1$

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2 (a) $Col(A)$ is spanned by the ~~rows~~ columns of A with the pivot elements, i.e., the first, third and fourth columns.

$$\Rightarrow Col(A) = Span\left\{ \begin{bmatrix} 2 \\ 6 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

(b) $Row(A) = Row(U) \Rightarrow$ the ~~three~~ non-zero rows of U form a basis for $Row(A)$

$$Row(A) = Span\{[2, 1, 0, 0, -1], [0, 0, -1, 0, 3], [0, 0, 0, 1, 1]\}$$

(c) $$A x = 0 \Leftrightarrow x = \begin{bmatrix} -x_2/2 + x_5/2 \\ x_2 \\ 3x_5 \\ -x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1/2 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} x_5$$

$$Nul(A) = Span\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(d) $Row(U) = Row(A)$ so the same answer as (b)

(e) $Col(A^t) = Row(A)$ —— " ——

(3) (a) We need to prove that $\mathbb{D}$ is a linearly independent set and that it spans H.

* Linear independence:

Suppose that $c_1(2+t^2)+c_2(t+t^2+2t^3)=0 \quad \forall t$

Then $2c_1 + c_2 t + (c_1+c_2)t^2 + 2c_2 t^3 = 0 \quad \forall t$

$\Rightarrow 2c_1=0, \; c_2=0, \; c_1+c_2=0, \; 2c_2=0 \Rightarrow c_1=c_2=0$

* To prove that $\mathbb{D}$ spans H, it is sufficient to prove that each element in the spanning set $S=\{4-t+t^2-2t^3, \; t+t^2+2t^3, \; -2+t+2t^3\}$ is spanned by the elements of $\mathbb{D}$.

$4-t+t^2-2t^3 = 2(2+t^2) - 1\cdot(t+t^2+2t^3)$

$t+t^2+2t^3 = 1\cdot(t+t^2+2t^3)$

$-2+t+2t^3 = -1\cdot(2+t^2) + 1\cdot(t+t^2+2t^3)$

(b) Set $d_1 = 2+t^2 \quad d_2 = t+t^2+2t^3$, then

$Ran(T) = Span\{T(d_1), T(d_2)\} = Span\{2t, 1+2t+6t^2\}$

Thus $G=\{2t, 1+2t+6t^2\}$ is a spanning set for $Ran(T)$, it just remains to verify that it is a linearly indep. set:

Assume that $c_1 2t + c_2(1+2t+6t^2)=0 \Rightarrow$

$\Rightarrow c_2 + 2(c_1+c_2)t + 6c_2 t^2 = 0 \Rightarrow c_2=0, \; 2(c_1+c_2)=0, \; 6c_2=0$

$\Rightarrow c_1=0 \;\&\; c_2=0$

(c) Since every non-zero element of H is a non-constant polynomial, only the zero element maps to zero.

$Nul(T)=\{0\}$

(4) (a) $Av_1 = \begin{bmatrix} 6-5+2-5 \\ 5-6+5-2 \\ 2-5+6-5 \\ 5-2+5-6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -2 \\ 2 \end{bmatrix} = (-2)\overbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}^{v_1} \Rightarrow \lambda_1 = -2$

$Av_2 = \begin{bmatrix} 6-5-2+5 \\ 5-6-5+2 \\ 2-5-6+5 \\ 5-2-5+6 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 4v_2 \Rightarrow \lambda_2 = 4$

(Remark: It is actually sufficient to compute the top row only!)

(b) Set $x = c_1 v_1 + c_2 v_2 \Rightarrow$

$\Rightarrow Ax = c_1 Av_1 + c_2 Av_2 = c_1\lambda_1 v_1 + c_2 \lambda_2 v_2 = -2c_1 v_1 + 4c_2 v_2$

Thus, $Ax = 2v_1 - v_2$ if $c_1 = -1$ & $c_2 = -\frac{1}{4}$

<u>Answer</u> $x = -v_1 - \frac{1}{4} v_2 = \begin{bmatrix} -5/4 \\ 5/4 \\ -3/4 \\ 3/4 \end{bmatrix}$

(c) $Bv_2 = \frac{1}{\lambda_2} v_2 \Rightarrow B^2 v_2 = \frac{1}{\lambda_2^2} v_2 = \frac{1}{16} v_2 = \begin{bmatrix} 1/16 \\ -1/16 \\ -1/16 \\ 1/16 \end{bmatrix}$

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(5) $PP^t = \begin{bmatrix} I & -Q^t \\ Q & I \end{bmatrix}\begin{bmatrix} I & Q^t \\ -Q & I \end{bmatrix} = \begin{bmatrix} I+Q^tQ & Q^t-Q^t \\ Q-Q & QQ^t+I \end{bmatrix} = \begin{bmatrix} 2I & 0 \\ 0 & 2I \end{bmatrix} = 2I_6$ ← 6×6 identity matrix.

Thus $P^{-1} = \frac{1}{2}P^t$. We use P to construct the similar matrix

$B = P^{-1}AP = \frac{1}{2}\begin{bmatrix} I & Q^t \\ -Q & I \end{bmatrix}\begin{bmatrix} I & 2Q^t \\ 2Q & I \end{bmatrix}\begin{bmatrix} I & -Q^t \\ Q & I \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I+2Q^tQ & 2Q^t+Q^t \\ -Q+2Q & -2QQ^t+I \end{bmatrix}\begin{bmatrix} I & -Q^t \\ Q & I \end{bmatrix} =$

$= \frac{1}{2}\begin{bmatrix} 3I & 3Q^t \\ Q & -I \end{bmatrix}\begin{bmatrix} I & -Q^t \\ Q & I \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3+3Q^tQ & -3Q^t+3Q^t \\ Q-Q & -QQ^t-I \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6I & 0 \\ 0 & -2I \end{bmatrix} = \begin{bmatrix} 3I & 0 \\ 0 & -I \end{bmatrix}$

Since A and B are similar, they have the same eigenvalues. Since B is diagonal, we simply read off that $\lambda = 3$ & $\lambda = -1$ are the eigenvalues.