Math 222a: Second Midterm Exam
7.00-8.15pm, Nov. 9, 2004.
Closed books, no notes, no calculators.

Question 1: No motivation is required for these questions (4p each):
(There is no relation between the A's in the five questions.)

(a) Is the set $H = \{[x_1, x_2] : x_2 \geq 0\}$ a subspace of $\mathbb{R}^2$?
(b) A is a 4×6 matrix such that the map $x \mapsto Ax$ is onto $\mathbb{R}^4$. What is $\mathrm{Col}(A)$?
(c) A is a matrix such that $\det(A - \lambda I) = 2\lambda - 4\lambda^2 + 2\lambda^3$. What are the eigenvalues of A?
(d) A is an $m \times n$ matrix of rank k. What is $\dim(\mathrm{Nul}(A))$?
(e) A is a 3×3 matrix with eigenvalues $1, 2, 3$ and I is the 3×3 identity matrix. What is $\dim(\mathrm{Nul}(A - 3I))$?

Question 2: The matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 \\ 6 & 3 & -1 & 0 & 0 \\ -2 & -1 & -1 & 1 & 5 \\ 4 & 2 & 0 & -2 & -4 \end{bmatrix},$$

is row equivalent to the matrix

$$U = \begin{bmatrix} 2 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Give a basis for each of the following five spaces (4p each):

(a) $\mathrm{Col}(A)$
(b) $\mathrm{Row}(A)$
(c) $\mathrm{Nul}(A)$
(d) $\mathrm{Row}(U)$
(e) $\mathrm{Col}(A^{\mathrm{t}})$

Give a (brief!) motivation for each answer.

Question 3: Consider the vectorspace $\mathbb{P}^3$ (the space of all polynomials of degree at most 3) and its subspace

$$H = \mathrm{Span}\{4 - t + t^2 - 2t^3,\ t + t^2 + 2t^3,\ -2 + t + 2t^3\}.$$

(a) Prove that $\mathcal{D} = \{2 + t^2,\ t + t^2 + 2t^3\}$ is a basis for H. (10p)
(b) The linear transformation $T : H \to \mathbb{P}^3$ is defined by

$$T : c_0 + c_1 t + c_2 t^2 + c_3 t^3 \quad \mapsto \quad c_1 + 2c_2 t + 3c_3 t^2.$$

Determine a basis for $\mathrm{Ran}(T)$. (5p)
(c) What is $\mathrm{Nul}(T)$? (5p)

Question 4: The invertible matrix

$$A = \begin{bmatrix} 6 & 5 & 2 & 5 \\ 5 & 6 & 5 & 2 \\ 2 & 5 & 6 & 5 \\ 5 & 2 & 5 & 6 \end{bmatrix}$$

has the eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

(a) Determine the eigenvalues λ_1 and λ_2 associated with v_1 and v_2. (5p)
(b) Find an x such that $Ax = 2v_1 - v_2$. (8p)
(c) Let $B = A^{-1}$. Determine $B^2 v_2$. (7p)

Question 5: (20p) Let Q be a 3×3 matrix such that $Q\,Q^{\mathrm{t}} = Q^{\mathrm{t}}\,Q = I$, where I is the 3×3 identity matrix. Determine the eigenvalues of the matrix

$$A = \begin{bmatrix} I & 2Q^{\mathrm{t}} \\ 2Q & I \end{bmatrix}.$$

Hint 1: Compute $P\,P^{\mathrm{t}}$, where

$$P = \begin{bmatrix} I & -Q^{\mathrm{t}} \\ Q & I \end{bmatrix}.$$

Hint 2: Can you use P in a similarity transform?