Math 222, Fall 2004: Notes on the final examination

General comments:

It is important that you know the properties of triangular, diagonal and orthogonal matrices. This is essential in order to understand the utility of the different matrix factorizations that we have discussed (LU, QR, SVD, eigenvalue-decomposition of symmetric matrices). As an example, if you're told that $A = LU$ and are given the factors L and U, you should know how to use them to solve the equation $A\mathbf{x} = \mathbf{b}$, how to determine what the rank of A is, how to get a basis for $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$, and so on.

When you answer exam questions, make sure to give (brief) explanations of the steps that you take. Even if your thinking is correct, you may lose points if I cannot follow your argument. As an example, if you're asked to prove that a set of polynomials is linearly independent, and you choose to prove it using Gaussian elimination, the answer should not consist merely of a sequence of row equivalent matrices. You need to first write something like "I form the matrix A whose columns are the coordinate vectors of the polynomials in the standard basis. The polynomials are linearly independent if and only if these columns are linearly independent. I will prove this by computing the echelon form of A." (Note however that no questions will require pages of text in the answer.)

None of the numerical notes is included. They contain very useful information though so I strongly recommend that you read them anyway.

Notes on the individual chapters:

When I say that something is not "core" material (or something to that effect), what I mean is that it can only come up in the final as either a hard question at the end, or as a small sub-question.

Chapter 1:

This whole chapter is about Gaussian elimination. Given a system of linear equations $A\mathbf{x} = \mathbf{b}$, you should know how to determine whether the system has a solution, and if it does, be able to write down a formula for all solutions.

Note that sections 1.6 and 1.10 are not included in the final.

Chapter 2:

Matrix arithmetics is extremely important, see Theorems 1,2,3 and the warnings on page 114.

Matrix inverse. When does it exist? Theorem 6 is important.

You should know how to USE the LU-factorization, see note above. The material on elementary matrices and the description of how to construct the LU factorization is included to the extent that it was covered in class but is not core material.

Same thing with partitioned matrices. The concept is very useful, there may be questions on it, but it is not core material.

Chapter 4:

All material in sections 4.1 - 4.7 is included.

The connection between the different spaces defined by a matrix (column / row / null space) and solvability of systems of equations is important.

When answering questions on this material, make particularly sure that you explain what you are doing.

Chapter 5:

See the detailed notes below on what is included.

It is very useful to know that if $\mathbf{v}$ is an eigenvector with eigenvalue λ, then $\mathbf{v}$ is an eigenvector for A^k corresponding to the eigenvalue λ^k, and for A^{-1} corresponding to the eigenvalue λ^{-1}.

Section 5.4 says something very remarkable: If an $n \times n$ matrix A has a set of eigenvectors that forms a basis for $\mathbb{R}^n$, then in the coordinate system defined by the eigenvectors, the representation of the matrix A is diagonal. This observation is particularly useful when considering symmetric matrices since we know that *all* symmetric matrices have a set of eigenvectors that forms a basis. Moreover, you don't have to settle for any old basis since you're assured that there exists an orthonormal basis.

Chapter 6:

The main point here is to show how geometric concepts like orthogonality and length generalize from $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$.

The concept of orthogonal complement is important. Spend some time thinking about the fact that $\mathrm{Nul}(A)$ is the orthogonal complement of $\mathrm{Row}(A)$.

Suppose that the subspace H of $\mathbb{R}^n$ has an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$. Then form the orthogonal matrix $U = [\mathbf{u}_1, \ldots, \mathbf{u}_k]$. Now, if $\mathbf{x}$ is a vector, you know that $\mathrm{proj}_H\mathbf{x} = U\,U^{\mathrm{t}}\mathbf{x}$. As a consequence, if you're given the singular value decomposition of a matrix, $A = U\,D\,V^{\mathrm{t}}$, you can very easily compute the orthogonal projection of a vector onto $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$. If you know the QR-decomposition, you can similarily easily project onto $\mathrm{Col}(A)$. Figure 4 in section 7.4 contains a lot of useful information.

The Gram-Schmidt process is included but the only question that can come up on the exam is some variation of the following: Given two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$, find an orthogonal basis for the set $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, or the same question with three vectors.

The book does not make it very clear that if you're given the matrices Q and R in the QR-factorization of A, then the columns of Q form an orthogonal basis for $\text{Col}(A)$, and the entries of R are simply the expansion coefficients of the columns in that basis. In other words, if $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, $Q = [\mathbf{q}_1, \ldots, \mathbf{q}_n]$, and the ij-entry of R is r_{ij}, then say the third column of A has the expansion $\mathbf{a}_3 = \mathbf{q}_1 r_{13} + \mathbf{q}_2 r_{23} + \mathbf{q}_3 r_{33}$.

Chapter 7:

The material of section 7.1 is exceptionally useful and if you intend to pursue science or engineering, it is very likely that you'll use these results over and over again.

Some of the material in section 7.2 is hard to understand unless you have a good understanding of multivariate calculus. If there is an exam question on this material, it will be formulated to minimize the need for calculus. The important point is that you understand the connection between quadratic forms and symmetrics matrices. Read the proof of Theorem 5 carefully.

/// PG Martinsson, Dec 7, 2004

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1.1: Everything. [7, 19-22, 25]
1.2: Everything. [1-20, 23-28]
1.3: Everything. [11-14, 17-22, 25, 26]
1.4: Everything. [1-10, 27, 28, 31, 32]
1.5: Everything. [1-14, 29-34]
1.7: Everything. [9-20, 23-30]
1.8: Everything. [17-20, 25, 31]
1.9: Everything. [25-28, 31-35]
2.1: Everything. [13, 17-26]
2.2: Everything. [11-24, 35]
2.3: Everything [3,5,15-24]
2.4: Everything [3,7,13,14,15]
2.5: Only material covered in class is included. In particular,
     the derivation of the LU factorization is not included. [1,5,7,9,21]
4.1: Everything [1-18,23,31]
4.2: Everything [3-6,17-25,31]
4.3: Everything [13,21,23,24,25]
4.4: Everything [1,5,13,15]
4.5: Everything [1,7,13,19,21]
4.6: Everything [3,5-17]
4.7: Everything [1,11,13]
5.1: Everything [1-21,27,29]
5.2: Regarding determinants; you should know formula (1),
     Theorem 3 and the formula for the 2 x 2 case.
     Also included are the concept of the characteristic
     polynomial (and its connection to eigenvalues), and
     the concept of "similarity". [1-8,15,17,21,23]
5.3: Everything except Theorem 7. Make sure you
     learn the four-step method described.
     [1-20,23-27]
5.4: Everything. [1,5,13,17,19-24]
6.1: Everything. [1-20,23,25,27,31]
6.2: Everything. [1-11,13,17-22,23,25,27,29]
6.3: Everything. [1-24]
6.4: Everything. (Thm 12 is more important than Thm 11.)
     [1,9,11,13,19,23]
6.5: Everything. [3,5,7,9,13,15,17,19,23,25]
7.1: Everything except the statements relating to multiplicity
     of eigenvalues. [1-30,32-36]
7.2: Read everything carefully. However, only material
     mentioned in class may appear on the final. [1-14,25,27]
7.3: Only the few facts mentioned in class are included.[3-7,9,11]
7.4: Only the material covered in class is included. In particular,
     you do not need to know how to construct the SVD. Examples
     6,7,8 are very useful. [5,6,11,12,15,17,19,21]
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