Power Iteration Methods

1. Numerically Stable Subspace Iteration

Let A be an $n \times n$ Hermitian matrix.

Draw $n \times l$ Gaussian vectors G.
$Y_1 = AG$
$Q_1 = \text{orth}(Y_1)$; `/* In this case, orth(A) is Q from QR decomposition. */`
$Y_2 = AQ_1$
$Q_2 = \text{orth}(Y_2)$
$\vdots$
$B = Q_k^* A Q_k$
$B = \hat{U} D \hat{U}$
$U = Q_k \hat{U}$
Then $A \approx UDU^*$.

This is a numerically stable version of the previously discussed algorithm.

Claim: Suppose $\dim(\text{col}(Q_j)) = \dim(A^j G) = l$ for $j = 1, 2, \ldots, q$. Then $\text{col}(Q_j) = \text{col}(A^j G)$.

Sketch of Proof: Trivial for $q = 1$. For $q = 2$,

$$Y_2 = AQ_1 = A \underbrace{Y_1}_{AG} R_1^{-1} = A^2 G R_1^{-1}$$

Since the dimensions are the same, $\text{col}(Q_2) = \text{col}(A^2 G)$. $q = 3$ follows in much the same way, and the rest is proved via induction.

Note: The assumption on dimensionality is unnecessary. If G is gaussian, and rank$(A) \geq l$, then the assumption holds with probability 1.

Note: This method is very conservative, and emphasizes numberical stability. Machine precision gets finicky if steps are skipped, so we have to consider the question of "good enough".

2. Diagonal Hermitian Matrices

"Every Hermitian matrix is 'morally' diagonal". What does this mean? Consider a 2×2 matrix A such that $A = A^*$ (definition of Hermitian). Then there exists an orthonormal basis $\{v_1, v_2\}$ of the eigenvectors of A.

$$A = \begin{bmatrix} \uparrow & \uparrow \\ v_1 & v_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \leftarrow v_1^* \rightarrow \\ \leftarrow v_2^* \rightarrow \end{bmatrix}$$

Let $x \in \mathbb{R}^2$, then

$$\begin{aligned} x &= v_1(v_1 \cdot x) + v_2(v_2 \cdot x) \\ &= v_1 v_1^* x + v_2 v_2^* x \\ &\Rightarrow \underbrace{\begin{bmatrix} \uparrow & \uparrow \\ v_1 & v_2 \\ \downarrow & \downarrow \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \leftarrow v_1^* \rightarrow \\ \leftarrow v_2^* \rightarrow \end{bmatrix}}_{V^*} x \end{aligned}$$

If we set

$$x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = V^* x$$

Then x_1', x_2' are the coordinates of x in the $\{v_1, v_2\}$ coordinate system.

$$\begin{aligned} y = Ax \Rightarrow V^* y &= V^* A V V^* X \\ &\Rightarrow y' = D x' \end{aligned}$$

Once you move into the coordinate system formed by $\{v_1, v_2\}$, the matrix is diagonal.

Note: all decompositions/operations are coordinate system independent.

3. Power Iteration for General Matrices

Let A be $m \times n$.
Let $A = UDV^*$ be the SVD of A.
$(AA^*)A = \underbrace{UDV^*}_{A}\underbrace{VDU^*}_{A}\underbrace{UDV^*}_{A} = UD^3V^*$.
$(AA^*)^2 A = (AA^*)(AA^*)A = UDV^*VDU^*UD^3V^* = UD^5V^*$.
etc: $(AA^*)^q A = UD^{2q+1}V^*$ (proved with induction).

The general idea of this algorithm is to start by drawing gaussian vectors, decompose, find Q, and continue.

$G = \text{randn}(n, l)$
$Y = AG$
for $j = 1, 2, \ldots, q$ **do**
  $Z = A^* Y$
  $Y = AZ$
end
$Q = \text{orth}(Y)$

This algorithm is the quick and dirty version that is great for fast approximation.

A slower, but more stable and accurate version is as follows:

$G = \text{randn}(n, l)$
$Y = AG$
$Q = \text{orth}(Y)$
for $j = 1, 2, \ldots, q$ **do**
  $Z = A^* Q$
  $W = \text{orth}(Z)$
  $Y = AW$
  $Q = \text{orth}(Y)$
end

4. Krylov Methods

That power iteration looks similar to another set of iterative methods, called Krylov methods.

Recall the single method power scheme for a square matrix.

$g = \text{rand}(n, 1)$
for $i = 1, 2, \ldots, p$ **do**
| $y_i = Ay_{i-1}$
end
Then $V_1 \approx y_p / \|y_p\|$

This algorithm is very wasteful however, as we lose all info on y_i. Consider the subspace $\mathcal{K} = \mathcal{K}(A, g) = \text{span}\{g, Ag, A^2g, \ldots, A^{p-1}g\}$. In a krylov method we project A onto $\mathcal{K}_p$ and use the eigenvalues of the resulting smaller matrix as approximations to the eigenvalues of A.

To be precise, set

$$Q = \text{orth}\left(\begin{bmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ g & Ag & A^2g & \ldots & A^{p-1}g \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}\right)$$

Set $T = Q^*AQ$. Using the Eigenvalue Decomposition of $T = \hat{U}D\hat{U}^*$, $U = Q\hat{U}$, therefore $A \approx UDU^*$.