APPM4720/5720 — Homework 4

Problem 4.1: Define a contour Γ_1 via

$$\Gamma_1 = \{x = (x_1,\, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + 2\,x_2^2 = 1\}.$$

Let Ω_1 denote the domain interior to Γ_1. Define points $a, b \in \Gamma_1$ and $c \in \Omega_1$ via

$$a = (1,\, 0), \qquad b = (\cos(0.7),\, (1/\sqrt{2})\,\sin(0.7)), \qquad c = (0.3,\, 0.2).$$

Let u be the unique solution to

$$(1) \qquad \begin{cases} -\Delta u(x) = 0, & x \in \Omega_1, \\ \quad u(x) = f(x), & x \in \Gamma_1, \end{cases}$$

where

$$f(x_1,\, x_2) = x_1\, e^{\sin(10\, x_2)}.$$

Let u have the representation

$$u(x) = [S\,\sigma](x) = \int_{\Gamma_1} \frac{1}{2\pi}\, \log \frac{R}{|x - x'|}\, \sigma(x')\, dl(x'),$$

where R is chosen so that $R/|x - x'| \geq 2$ for all $x,\, x' \in \Gamma$. (Say $R = 10$.)

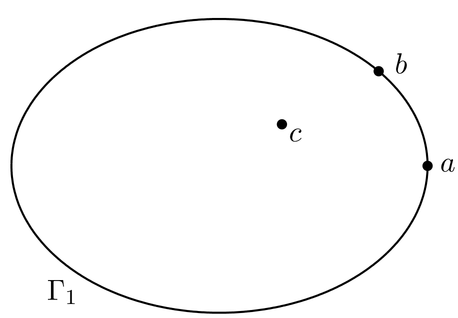
Your task is to form an equation for σ, discretize this equation, solve the equation, and then to evaluate the function u.

Let N denote the number of degrees of freedom in your approximation, and include in your solution the following table (with values filled in where the question marks are):

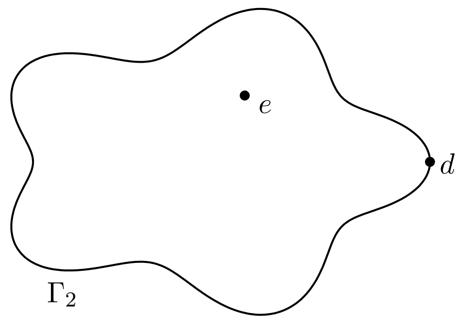
N	$\sigma(a)$	$\sigma(b)$	$u(c)$
100	?	?	?
200	?	?	?
400	?	?	?
800	?	?	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Include as large N as your computer can handle in a reasonable amount of time, and estimate the convergence rate for each column.

Hint: In developing the code, it might be helpful to solve some Laplace problems for which you *know* the solution. For instance, set $v(x_1,\, x_2) = x_1^2 - x_2^2$ and $f = v(\Gamma)$. Or, pick a point z outside of Γ, and set $v(x) = \log|x - z|$. (If you try the latter option, what happens if you pick z very close to Γ?)



The geometry of Problems 4.1 and 4.3.



The geometry of Problems 4.2 and 4.4.

Problem 4.2: Repeat Problem 4.1, but now set

$$G_1(t) = 1.5\,\cos(t) + 0.1\,\cos(6\,t) + 0.1\,\cos(4\,t),$$
$$G_2(t) = \sin(t) + 0.1\,\sin(6\,t) - 0.1\,\sin(4\,t),$$

and define

$$\Gamma_2 = \{x = (G_1(t),\, G_2(t))\,:\, t \in [0,\, 2\pi)\}.$$

The Dirichlet data f is the same. Report $\sigma(d)$ and $u(e)$ for the points

$$d = (1.7,\, 0), \qquad e = (0.3,\, 0.5).$$

Problem 4.3: Repeat Problem 4.1 (with the contour Γ_1) but now use the Ansatz

$$u(x) = [D\sigma](x) = \int_{\Gamma_1} \frac{n(x')\cdot(x-x')}{2\pi|x-x'|^2}\,\sigma(x')\,dl(x')$$

where $n(x')$ is the outwards pointing unit normal at x'.

Problem 4.4: Repeat Problem 4.2 (with the contour Γ_2) but now use the Ansatz

$$u(x) = [D\sigma](x) = \int_{\Gamma_2} \frac{n(x')\cdot(x-x')}{2\pi|x-x'|^2}\,\sigma(x')\,dl(x')$$

where $n(x')$ is the outwards pointing unit normal at x'.

Hint: In debugging your codes for the double layer potential, you may find the third Green identity useful (in particular that $[D1](x) = -1$ when $x \in \Omega$, and $[D1](x) = -1/2$ when $x \in \Gamma$).

Hint: In my codes, I represent a contour Γ in an object `C` of size $6 \times N$, where N is the number of discretization points. Column `i` of `C` encodes the data for parameter point t_i:

$$\begin{aligned}
\texttt{C}(1,i) &= G_1(t_i)\\
\texttt{C}(2,i) &= G_1'(t_i)\\
\texttt{C}(3,i) &= G_1''(t_i)\\
\texttt{C}(4,i) &= G_2(t_i)\\
\texttt{C}(5,i) &= G_2'(t_i)\\
\texttt{C}(6,i) &= G_2''(t_i)
\end{aligned}$$