Hilbert-Schmidt operators

Review of $H = \mathbb{C}^n$ Let A be an $n \times n$ complex matrix.

Recall the def^n of the Frobenius norm of A:

$$\|A\|_F = \Big(\sum_{i,j=1}^{n} |a_{ij}|^2 \Big)^{1/2} = (\text{trace } A^*A)^{1/2}$$

Let us compare $\|\cdot\|_F$ to the standard norm $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

$$\|Ax\|^2 = \sum_{i=1}^{n} |r^{(i)} \cdot x|^2 \underset{\text{C-S}}{\leq} \sum_{i=1}^{n} |r^{(i)}|^2 |x|^2 = \|A\|_F^2 |x|^2$$

$$A = \begin{bmatrix} - r^{(1)} - \\ - r^{(2)} - \\ \vdots \\ - r^{(n)} - \end{bmatrix}$$

So $\|A\| \leq \|A\|_F$

Homework: (a) Show $\|A\| \leq \|A\|_F \leq \sqrt{n}\, \|A\|$

(b) Find matrices B & C s.t. $\|B\| = \|B\|_F$ [redacted]

$\sqrt{n}\, \|C\| = \|C\|_F$ [redacted]

Now let $(\varphi^{(j)})_{j=1}^{n}$ be an ON-basis for H.

We have $\sum_{j=1}^{n} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} |r^{(i)} \cdot \varphi^{(j)}|^2 = \sum_{i=1}^{n} |r^{(i)}|^2 = \|A\|_F^2$

Alternative def^n: Let $(\varphi^{(j)})_{j=1}^{n}$ be an ON-basis. Set $\|A\|_F = \Big(\sum_{j=1}^{n} \|A\varphi^{(j)}\|^2 \Big)^{1/2}$

Now suppose A has an ON-basis such that $A\varphi^{(n)} = \lambda_n \varphi^{(n)}$.

Then $\|A\|_F^2 = \sum_{j=1}^{n} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{n} |\lambda_j|^2$

General Hilbert Space

Lemma Let H be a H.S. and let $A \in \mathcal{B}(H)$.
Let (φ_j) & (ψ_j) be ON-bases for H.
Then $\sum_{j=1}^{\infty} \|A\varphi_j\|^2 = \sum_{j=1}^{\infty} \|A\psi_j\|^2$. (Either both are infinite, or they are both finite and identical.)

Def^n If for some ON-basis (φ_j) it is the case that $\sum \|A\varphi_j\|^2 < \infty$, then we say that H is a Hilbert-Schmidt operator, and define
$$\|A\|_{H\text{-}S} = \left(\sum_{j=1}^{\infty} \|A\varphi_j\|^2\right)^{1/2}.$$

Note: The def^n does not depend on the choice of basis!

Claim Every H-S operator is compact, but not every compact operator is H.S.

Lemma $\|A\| \leq \|A\|_{H\text{-}S}$.

Lemma If H has an ON-basis (φ_j) s.t. $A\varphi_j = \lambda_j \varphi_j$, then
$$\|A\|_{H\text{-}S} = \left(\sum_{j=1}^{\infty} |\lambda_j|^2\right)^{1/2}$$

Example $H = \ell^2$ $\quad (\lambda_j)_{j=1}^{\infty}$ is a seq in $\mathbb{C}$.

$Ax = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$

$(e_j)_{j=1}^{\infty}$ is an ON-basis for H s.t. $Ae_j = \lambda_j e_j$

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & - \\ 0 & \lambda_2 & 0 & - \\ 0 & 0 & \lambda_3 & - \\ | & | & | & \end{bmatrix}$$

~~Whe~~ We have: A is bdd $\iff \sup |\lambda_j| < \infty$

A is compact $\iff |\lambda_j| \to 0$ as $j \to \infty$

A is H-S $\iff \sum_{j=1}^{\infty} |\lambda_j|^2 < \infty$

Example $H = L^2(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$.

$$[Au](x) = \int_{\Omega} k(x,y)\, u(y)\, dA(y)$$

$\|A\|_{HS}^2 = \int_{\Omega}\int_{\Omega} |k(x,y)|^2 dA(y)$ so A is H-S $\iff k \in L^2(\Omega^2)$

If A is also S-A, then $\exists\, (\varphi_n)$ & (λ_n) s.t.

$$[Au](x) = \sum \lambda_n \varphi_n(x)(\varphi_n, u) = \sum \lambda_n \varphi_n(x) \int_{\Omega} \varphi_n(y)\, u(y)\, dA(y) =$$

$$= \int_{\Omega} \left(\sum \lambda_n \varphi_n(x) \overline{\varphi_n(y)} \right) u(y)\, dA(y)$$

so $\quad k(x,y) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \overline{\varphi_n(y)}$

FUNCTIONS OF OPERATORS

Let $f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_d z^d$

be a complex valued polynomial (α_j & $z \in \mathbb{C}$).

It is obvious how to define $f(A)$ for $A \in \mathcal{B}(H)$:

$$f(A) = \alpha_0 + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_d A^d$$

If A admits a spectral decomp $A = \sum_{n=1}^{\infty} \lambda_n P_n$

where $P_n P_m = 0$ if $n \neq m$ & $P_n^2 = P_n$ then $A^k = \sum_{n=1}^{\infty} \lambda_n^k P_n$.

It follows that $f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$.

Now let us generalize slightly to analytic functions.

Let $f(z)$ be analytic on $B_0(R)$, in other words, the sum

$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ converges absolutely for $|z| < R$.

Then if $A \in \mathcal{B}(H)$ is an operator s.t. $\|A\| < R$,

the sum $f_N(A) = \sum_{n=0}^{N} \alpha_n A^n$ converges in norm.

(You can easily prove that it is Cauchy.)

We define $f(A)$ as the limit: $f(A) = \sum_{n=0}^{\infty} \alpha_n A^n$.

<u>Example</u> $f(x) = (1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ for $|z| < R = 1$

$$f(A) = (I - A)^{-1} = \sum_{n=0}^{\infty} A^n \quad \text{for } \|A\| < R = 1$$

<u>Example</u> $f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $|z| < R = \infty$

$$f(A) = \exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \text{ for any } A \in \mathcal{B}(H).$$

If $A = \sum_{n=1}^{\infty} \lambda_n P_n$ and $r(A) = \sup|\lambda_n| < R$, then you can prove that $f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$.

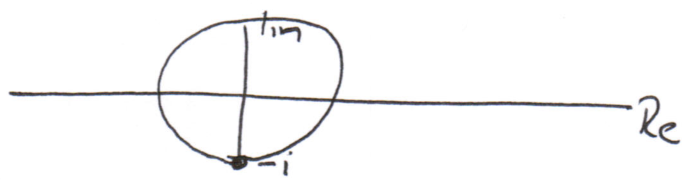
<u>HOWEVER</u>: The sum $f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$ may be norm convergent even when $r(A) > R$.

When it is, we use

$$\boxed{f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n}$$

as the definition of $f(A)$.

<u>Example</u> $f(z) = \frac{z-i}{z+i}$



The radius of analyticity is only 1, so the defⁿ via power series would work only when $\|A\| < 1$.

But if $A = \sum_{n=1}^{\infty} \lambda_n P_n$ and $|\lambda_n + i| > \delta$ for all n, then $f(A) = \sum_{n=1}^{\infty} \frac{\lambda - i}{\lambda + i} P_n$ is well defined.

~~with $\|f(A)\| \leq 1/\delta$ since~~

Note that if $A = \sum_{n=1}^{\infty} \lambda_n P_n$ and $\lambda_n \in \mathbb{R}$ (i.e. A is S-A), then $\left|\frac{\lambda_n - i}{\lambda_n + i}\right| = 1$ so A is unitary if $\sum_{n=1}^{\infty} P_n = I$.

More generally it can be shown that if A is <u>any</u> self-adjoint operator, then $B = (A - iI)(A + iI)^{-1}$ is unitary. (This conversion is known as a Cayley transform.)

General theory (for orientation only).

If A is a normal bounded operator, then A admits a spectral decomposition

$$A = \int_{\sigma(A)} \lambda \, dP(\lambda)$$

where P is a "projection valued measure".

The special case $A = \sum_{n=1}^{\infty} \lambda_n P_n$ occurs when ~~all~~ all of $\sigma(A)$ consists of eigenvalues (with possibly the additional point 0 in $\sigma_c(A)$.)

If f is continuous & bdd on $\sigma(A)$ we define

$$f(A) = \int_{\sigma(A)} f(\lambda) \, dP(\lambda).$$

Then $f(\sigma(A)) = \sigma(f(A))$.

Note: $f(\sigma(A)) \subseteq \{z : |z|=1\} \Rightarrow f(A)$ is unitary

$f(\sigma(A)) \subset \mathbb{R} \Rightarrow f(A)$ is S-A

$\mathrm{Re}(f(\sigma(A))) = 0 \Rightarrow f(A)$ is skew-adjoint.

$f(A)$ is always normal.