UNITARY MAPS – A GENERALIZATION OF ORTHOGONAL MATRICES

Def^n Let H_1 & H_2 be two Hilbert Spaces.
An operator $U : H_1 \to H_2$ is said to be UNITARY if it is bijective, and if

$$(Ux, Uy)_{H_2} = (x,y)_{H_1} \qquad \forall x,y \in H_1$$

In other words, a unitary map is a Hilbert Space Isomorphism.

What happens if $H_1 = H_2$?

U is unitary $\iff (Ux, Uy) = (x,y) \quad \forall x,y$
$\iff (U^*Ux, y) = (x,y) \quad \forall x,y$

So if U is unitary, then $U^*U = I$.
However, it is not the case that $U^*U = I \Rightarrow U$ unitary since U must also be bijective.

Counterexample Let R denote the right-shift operator.
R is one-to-one, but not onto.
However, $R^*R = I$.

So we must require both that U is invertible, and that $U^*U = I$.

Lemma Let H be a H.S. and let $U \in B(H)$.
Then U is unitary $\iff U$ is invertible, and $U^{-1} = U^*$.

Note that if $U \in B(H)$ is unitary, then both $UU^* = I$ and $U^*U = I$
(this is not true for the right-shift operator!)

<u>Example</u> $H = \ell^2(\mathbb{Z})$ ← doubly infinite sequences.

so $x \in H \iff x = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ and $\sum_{j=-\infty}^{\infty} |x_j|^2 < \infty$

Let R denote the rightshift operator on H.

$$[Rx]_j = x_{j-1}$$

Then $R^* = L$, the left-shift operator, and $R^{-1} = R^* = L$

so R <u>is</u> a unitary operator.

<u>Example</u> Let A be a bdd S-A operator on a H.S. H.

Set $B = \exp(iA) = \sum_{n=0}^{\infty} \frac{1}{n!} (iA)^n$

Then $B^* = \left(\sum_{n=0}^{\infty} \frac{1}{n!} (iA)^n \right)^* = \sum_{n=0}^{\infty} \frac{1}{n!} (-iA^*)^n$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (-iA)^n = \exp(-iA) = B^{-1}$$

Use $A^* = A$

So B is a unitary map.

(Recall that if $\lambda \in \mathbb{R}$ then $|e^{i\lambda}| = 1$ which is the analogous result in the 1-dim Hilbert space $\mathbb{C}$.)

<u>Example</u> Let H_1 be a H.S. with an ON-basis $(\phi_n)_{n=-\infty}^{\infty}$.
Let H_2 be a H.S. with an ON-basis $(\psi_n)_{n=-\infty}^{\infty}$
Then the map $U : H_1 \to H_2 : x \mapsto \sum_{n=-\infty}^{\infty} (\phi_n, x)\psi_n$
is a unitary map.
This is the "change of coordinate" map.
For instance, the Fourier transform
is of this type with

$H^1 = L^2(\mathbb{T})$ $\qquad \phi_n = \dfrac{e^{int}}{\sqrt{2\pi}}$

$H^2 = \ell^2(\mathbb{Z})$ $\qquad \psi_n = (\dots 0, 0, \overset{n\text{'th position}}{\overset{\downarrow}{1}}, 0, 0, \dots)$

$f \in L^2(\mathbb{T}) \Rightarrow Uf = (\alpha_n)_{n=-\infty}^{\infty}$

where $\alpha_n = (\phi_n, f) = \displaystyle\int_{\mathbb{T}} \frac{e^{-int}}{\sqrt{2\pi}} f(t)\,dt$

NORMAL OPERATORS

Defⁿ Let H be a H.S. An operator $N \in \mathcal{B}(H)$ is NORMAL if $NN^* = N^*N$.

Thm Let $H = \mathbb{C}^n$ and let A be an $n \times n$ complex valued matrix. Then:
A is normal $\iff$ $\exists$ diagonal D and unitary U such that $A = UDU^*$

So normal matrices are the matrices for which there is an ON basis $\{u_j\}_{j=1}^n$ of eigenvectors.

Let A be an $n \times n$ matrix of the form $A = UDU^*$ with U unitary and $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ diagonal. Then:

* A is self-adjoint $\iff A = A^* \iff \mathrm{Im}(\lambda_j) = 0 \quad \forall j$
* A is skew-adjoint $\iff A = -A^* \iff \mathrm{Re}(\lambda_j) = 0 \quad \forall j$
* A is unitary $\iff A^{-1} = A^* \iff |\lambda_j| = 1 \quad \forall j$

All three are special cases of normal matrices.

WEAK CONVERGENCE IN A HILBERT SPACE

Recall that if $\mathfrak{X}$ is a BANACH SPACE, then

$$x_n \rightharpoonup x \iff \varphi(x_n) \to \varphi(x) \quad \forall \varphi \in \mathfrak{X}^*.$$

"(x_n) converges weakly to x"

In a Hilbert space, we know that every functional φ takes the form $\varphi(x) = \langle y, x \rangle$ for some $y \in H$.

Consequently, in a H.S., we have

$$x_n \rightharpoonup x \iff \langle y, x_n \rangle \to \langle y, x \rangle \quad \forall y \in H.$$

<u>Example</u> $\mathfrak{X} = \ell^2(\mathbb{N})$, $x_n = e_n$,

For any $y = (y_1, y_2, y_3, \dots)$ we have

$$\langle y, e_n \rangle = y_n$$

and since $\sum |y_n|^2 < \infty$, we must have $e_n \rightharpoonup 0$.

<u>Example</u> $\mathfrak{X} = L^2(\mathbb{T})$ and $e_n(x) = \dfrac{e^{inx}}{\sqrt{2\pi}}$.

Then $e_n \rightharpoonup 0$.

<u>Observation:</u> In both examples, we have $\|e_n\| = 1$.

This means that just because $x_n \rightharpoonup x$, it is <u>not</u> necessarily the case that $\|x_n\| \to \|x\|$.

Recall from the Banach space case that

$$x_n \rightharpoonup x \implies \|x\| \leq \liminf_{n \to \infty} \|x_n\| \qquad (1)$$

The relation (1) follows from the Hahn-Banach thm:

$$\|x\| \overset{H\text{-}B}{=} \sup_{\|\varphi\|=1} |\varphi(x)| = \sup_{\|\varphi\|=1} \lim_{n\to\infty} |\varphi(x_n)| \le$$

$$\le \liminf_{n\to\infty} \sup_{\|\varphi\|=1} |\varphi(x_n)| = \liminf_{n\to\infty} \|x_n\|.$$

Further, we recall from the Banach space case that any weakly convergent sequence is bounded. This is ~~#~~ a consequence of the following thm:

THEOREM (Banach-Steinhaus) Let $\mathfrak{X}$ be a Banach space. Suppose $(\varphi_n)_{n=1}^{\infty}$ is a sequence in $\mathfrak{X}^*$, and that for every $x \in \mathfrak{X}$ we have

$$\sup_n |\varphi_n(x)| < \infty.$$

Then $\sup_n \|\varphi_n\| < \infty$.

The remarkable assertion here is that pointwise boundedness $\Rightarrow$ uniform boundedness

THEOREM Let (x_n) be a sequence in a Hilbert space H. Let Ω be a dense subset of H. Then:

$$x_n \rightharpoonup x \iff \begin{cases} (a) & \sup \|x_n\| < \infty \\ (b) & \langle y, x_n\rangle \to \langle y, x\rangle \quad \forall y \in \Omega \end{cases}$$

PROOF "$\Rightarrow$" Suppose that $x_n \rightharpoonup x$.

That (b) holds follows immediately from the def^n of weak convergence.

To prove (c), introduce the functionals $\varphi_n(y) = \langle x_n, y\rangle$.

Then $\sup_n |\varphi_n(y)| < \infty$ for every y since $\{\varphi_n(y)\}_{n=1}^{\infty}$ is convergent.

Banach-Steinhaus $\Rightarrow$ $\sup_n \|\varphi_n\|_{H^*} < \infty$.

Now just observe that $\|\varphi_n\|_{H^*} = \|x_n\|_H$.

"$\Leftarrow$" Suppose that (c) and (b) both hold.

Fix $z \in H$. We want to prove that $\langle x_n, z\rangle \to \langle x, z\rangle$.

Fix $\varepsilon > 0$ and pick $y \in \Omega$ s.t. $\|y - z\| < \varepsilon$.

Then
$$|\langle x_n - x, z\rangle| = |\langle x_n - x, y\rangle + \langle x_n - x, z - y\rangle|$$
$$\leq |\langle x_n - x, y\rangle| + \|x_n - x\|\,\|z - y\|$$
$$\leq |\langle x_n - x, y\rangle| + (M + \|x\|)\,\varepsilon$$
$\uparrow$ $M = \sup_n \|x_n\|$

Then
$$\limsup_{n\to\infty} |\langle x_n - x, z\rangle| \leq \underbrace{\limsup_{n\to\infty} |\langle x_n - x, y\rangle|}_{=0} + (M + \|x\|)\varepsilon$$

Since ε was arbitrary, $\limsup_{n\to\infty} |\langle x_n - x, z\rangle| = 0$.

AA2d (36c)

Example Let H be a H.S. and let $(\varphi_\alpha)_{\alpha\in A}$ be an ON-basis for H.

Set $\Omega = \text{span}\{\varphi_\alpha\}_{\alpha\in A}$

FINITE linear combinations.

Now suppose that $(x_n)_{n=1}^\infty$ is a sequence for which $M = \sup_n \|x_n\| < \infty$.

Suppose that you can also show that

$$\langle x_n, \varphi_\alpha\rangle \to \langle x, \varphi\rangle \qquad \forall \alpha\in A.$$

Then obviously

$$\langle x_n, y\rangle \to \langle x, y\rangle \qquad \forall y\in\Omega.$$

This means that $(x_n)_{n=1}^\infty$ is weakly convergent.

$$\boxed{\text{You just need to verify that the sequence is bounded, and that "each coordinate" } (\langle x_n,\varphi_\alpha\rangle)_{n=1}^\infty \text{ converges.}}$$

Prop^n Let H be a H.S.

(a) If $x_n \rightharpoonup x$, then $\|x\| \leq \liminf \|x_n\|$

(b) If $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

strong conv.!

Proof (a) This follows immediately from the Banach space result. However, it can be proven very easily for a H.S.:

$$\|x\|^2 = \lim_{n\to\infty} \langle x_n, x\rangle \leq \underset{n\to\infty}{\lim\overset{\inf}{\cancel{\sup}}} \|x_n\| \, \|x\|$$

(b) Suppose that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then

$$\|x - x_n\|^2 = \|x\|^2 - \langle x, x_n\rangle - \langle x_n, x\rangle + \|x_n\|^2$$

Observe that $\langle x, x_n\rangle \to \|x\|^2$ since $x_n \rightharpoonup x$. Then

$$\lim_{n\to\infty} \|x - x_n\|^2 = \|x\|^2 - \lim \langle x, x_n\rangle - \lim \langle x_n, x\rangle + \lim \|x_n\|^2$$

$$= \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0.$$

Recall the following thm for Banach spaces: <u>AA 2d</u>

Thm (Banach Alaoglu) Let X be a Banach space. Set $B = \{\varphi \in X^* : \|\varphi\| \leq 1\}$. Then B is compact in the weak-$*$ topology.

Recall that in a reflexive space, and in particular in Hilbert spaces, the weak and weak-$*$ topologies are the same.

Corollary Let H be a H.S.
Set $B = \{x \in H : \|x\| \leq 1\}$.
Then B is compact in the weak topology.

Corollary Let H be a ~~sequence~~ Hilbert space, and let $(x_n)_{n=1}^{\infty}$ be a seq. such that $\sup_n \|x_n\| < \infty$.
Then $\exists$ a subseq $(x_{n_j})_{j=1}^{\infty}$ such that $x_{n_j} \rightharpoonup x$ for some $x \in H$.

"Every bdd seq has a weakly convergent subseq."