APPM5450 — Applied Analysis: Section exam 3
1:00pm – 1:40pm, April 28, 2017. Closed books.

The in-class part of the exam consists of Problems 1 – 4. ***This needs to be submitted by 1:40pm to allow time for FCQs!*** No motivations necessary for any of these problems except 4(e).

Problem 1: (6p) Consider the function $f_n(x) = \big(1/x\big)^{1/3}\,\chi_{[1,n]}$ so that

$$f_n(x) = \begin{cases} x^{-1/3} & \text{when } x \in [1,n], \\ 0 & \text{when } x \notin [1,n]. \end{cases}$$

For which $p \in [1,\infty]$ does $(f_n)_{n=1}^{\infty}$ form a Cauchy sequence in $L^p(\mathbb{R})$?

Problem 2: (7p) Define for $n = 1,\,2,\,3,\,\dots$ the functions $f_n = \chi_{[-n,n]}$ so that

$$f_n(x) = \begin{cases} 1 & \text{when } x \in [-n,n], \\ 0 & \text{when } x \notin [-n,n]. \end{cases}$$

(a) (4p) Specify the Fourier transform $\hat{f}_n$.

(b) (3p) Consider the sequence $(\hat{f}_n)_{n=1}^{\infty}$. Specify its limit point in $\mathcal{S}^*(\mathbb{R})$.

Problem 3: (9p) Consider the function $f(x) = e^{-x^2/2}$ as a member of $L^2(\mathbb{R})$. Recall that its Fourier transform is $\hat{f}(t) = e^{-t^2/2}$.

(a) (3p) Set $g(x) = e^{-(x-1)^2/2}$. Specify $\hat{g}$.

(b) (3p) Set $h(x) = x\,e^{-x^2/2}$. Specify $\hat{h}$.

(c) (3p) Set $k(x) = e^{-x^2}$. Specify $\hat{k}$.

Problem 4: (18p) Let $p, q \in [1,\infty)$. Set $I = [0,1]$. Circle the correct answer. No penalties for guessing (so 3p for correct answer, 0p for incorrect or no answer).

(a) (3p) If $p < q$, then it is necessarily the case that $L^p(\mathbb{R}) \subseteq L^q(\mathbb{R})$. TRUE / FALSE.

(b) (3p) If $q < p$, then it is necessarily the case that $L^p(\mathbb{R}) \subseteq L^q(\mathbb{R})$. TRUE / FALSE.

(c) (3p) If $p < q$, then it is necessarily the case that $L^p(I) \subseteq L^q(I)$. TRUE / FALSE.

(d) (3p) If $q < p$, then it is necessarily the case that $L^p(I) \subseteq L^q(I)$. TRUE / FALSE.

(e) (6p) Provide on a separate sheet motivations for two of your answers (pick any two).

Solutions to problems 5, 6, and 7 are due on Wednesday May 3 by 5pm. Open books.

Problem 5: (10p) Provide a motivation for your answer to Problem 1.

Problem 6: (10p) Provide motivations for your answers to Problem 2.

Problem 7: (20p) Let Ω be an interval in $\mathbb{R}$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued measurable functions on Ω that converges *pointwise.* In other words, there is a function f such that

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \forall x \in \Omega.$$

Let $g \in L^2(\Omega)$, and define, whenever the integral exists,

$$\alpha_n = \int_\Omega \frac{f_n(x)}{1 + |f_n(x)|}\, g(x)\, dx. \tag{1}$$

(a) (10p) Let $\Omega = [0, 1]$. Prove that the integral in (1) is a well-defined Lebesgue integral that evaluates to a finite number α_n, and that

$$\lim_{n\to\infty} \alpha_n = \int_\Omega \frac{f(x)}{1 + |f(x)|}\, g(x)\, dx.$$

(b) (10p) Let $\Omega = \mathbb{R}$. Provide examples of functions (f_n) and g such that (1) is well-defined as a Lebesgue integral for every n, but so that the limit of (α_n) either does not exist, or does not equal $\int_\Omega \frac{f(x)}{1+|f(x)|}\, g(x)\, dx$.