

## Homework set 9 — APPM5450, Spring 2014

From the textbook: 11.5, 11.9, 11.15.

**Problem 1:** We say that a sequence  $(\varphi_n)_{n=1}^{\infty}$  is an *approximate identity* if

- (1)  $\varphi_n \in C(\mathbb{R}^d)$ ,  $\forall n$ ,
- (2)  $\varphi_n(x) \geq 0$ ,  $\forall n, x$ ,
- (3)  $\int_{\mathbb{R}^d} \varphi_n(x) dx = 1$ ,  $\forall n$ ,
- (4)  $\forall \varepsilon > 0$ ,  $\int_{|x| \geq \varepsilon} \varphi_n(x) dx \rightarrow 0$  as  $n \rightarrow \infty$ .

(a) Do the conditions imply that  $\varphi_n \in \mathcal{S}^*$ ?

(b) Assuming that  $\varphi_n \in \mathcal{S}^*$ , prove that  $\varphi_n \rightarrow \delta$  in  $\mathcal{S}^*$ .

**Problem 2:** Compute the Fourier transforms of  $f(x) = \chi_{[-R,R]}(x)$  and  $f(x) = e^{-a|x|}$  by simply evaluating the formula

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} f(x) dx.$$

The answers are given in examples 11.32 and 11.33 in the text book.

**Problem 3 (optional):** Let  $k$  be a positive integer. Prove that there exist numbers  $c_k$  and  $C_k$  such that  $0 < c_k \leq C_k < \infty$ , and

$$(1) \quad c_k (1 + |x|^k) \leq (1 + |x|^2)^{k/2} \leq C_k (1 + |x|^k), \quad \forall x \in \mathbb{R}^d.$$

Check to see if you can readily adapt your proof to also prove the existence of numbers  $b_k$  and  $B_k$  such that  $0 < b_k \leq B_k < \infty$  such that

$$(2) \quad b_k (1 + |x|)^k \leq (1 + |x|^2)^{k/2} \leq B_k (1 + |x|)^k, \quad \forall x \in \mathbb{R}^d.$$

*Note 1:* The existence of inequalities such as (1) and (2) are routinely used (generally without even commenting on it) to replace the growth factor  $(1 + |x|^2)^{k/2}$  in the norms  $\|\cdot\|_{\alpha,k}$  by either  $(1 + |x|^k)$  or  $(1 + |x|)^k$ , whenever convenient.

*Note 2:* If you have time, you may find it interesting to see what happens to the numbers  $b_k$ ,  $B_k$ ,  $c_k$ ,  $C_k$  as  $k$  grows large. (This is easily done using Matlab.)