Hints for homework set 8 — APPM5450, Spring 2017

11.5: Note that
$$\frac{1}{x + i\varepsilon} = \frac{x}{\varepsilon^2 + x^2} - i\frac{\varepsilon}{\varepsilon^2 + x^2}.$$

Fix a $\varphi \in \mathcal{S}$. You need to prove that

(1)
$$\lim_{\varepsilon \to 0}\langle i\frac{\varepsilon}{\varepsilon^2 + x^2}, \varphi\rangle \to -i\pi\varphi(0).$$

and that

(2)
$$\lim_{\varepsilon \to 0}\langle\frac{x}{\varepsilon^2 + x^2}, \varphi\rangle \to \langle\mathrm{PV}\left(\frac{1}{x}\right), \varphi\rangle,$$

Proving (1) is simple:
$$\langle i\frac{\varepsilon}{\varepsilon^2 + x^2}, \varphi\rangle = \int_{-\infty}^{\infty} i\frac{\varepsilon}{\varepsilon^2 + x^2}\varphi(x)\,dx = \{\text{Set } x = \varepsilon y\} = \cdots$$

For (2) we need to work a bit more (unless I overlook a simpler solution)

$$\lim_{\varepsilon \to 0}\langle\frac{x}{\varepsilon^2 + x^2}, \varphi\rangle - \langle\mathrm{PV}\left(\frac{1}{x}\right), \varphi\rangle$$

$$= \lim_{\varepsilon \to 0}\int_{-\infty}^{\infty}\frac{x}{\varepsilon^2 + x^2}\varphi(x)\,dx - \lim_{\varepsilon \to 0}\int_{|x| \geq \sqrt{\varepsilon}}\frac{1}{x}\varphi(x)\,dx$$

$$= \underbrace{\lim_{\varepsilon \to 0}\int_{|x| \geq \sqrt{\varepsilon}}\left(\frac{x}{\varepsilon^2 + x^2} - \frac{1}{x}\right)\varphi(x)\,dx}_{=S_1} + \underbrace{\lim_{\varepsilon \to 0}\int_{|x| \leq \sqrt{\varepsilon}}\frac{x}{\varepsilon^2 + x^2}\varphi(x)\,dx}_{=S_2}.$$

First we bound $|S_1|$. Note that when $|x| \geq \sqrt{\varepsilon}$, we have
$$\left|\frac{x}{\varepsilon^2 + x^2} - \frac{1}{x}\right| = \frac{\varepsilon^2}{|x|(\varepsilon^2 + x^2)} \leq \frac{\varepsilon^2}{|x|^3} \leq \frac{\varepsilon^2}{\varepsilon^{3/2}} = \sqrt{\varepsilon}.$$

Consequently,

$$|S_1| \leq \limsup_{\varepsilon \to 0}\int_{|x| \geq \sqrt{\varepsilon}}\left|\frac{x}{\varepsilon^2 + x^2} - \frac{1}{x}\right||\varphi(x)|\,dx$$

$$\leq \limsup_{\varepsilon \to 0}\int_{|x| \geq \sqrt{\varepsilon}}\sqrt{\varepsilon}\frac{1}{(1 + |x|^2)}\underbrace{|(1 + |x|^2)\varphi(x)|}_{\leq \|\varphi\|_{0,2}}\,dx = 0.$$

In bounding S_2 we use that
$$\int_{|x| \leq \sqrt{\varepsilon}}\frac{x}{\varepsilon^2 + x^2}\varphi(0)\,dx = 0,$$

and that
$$|\varphi(x) - \varphi(0)| \leq |x|\,\|\varphi'\|_{\mathrm{u}} \leq |x|\|\varphi\|_{1,0},$$

to obtain

$$|S_2| = \left|\lim_{\varepsilon \to 0}\int_{|x| \leq \sqrt{\varepsilon}}\frac{x}{\varepsilon^2 + x^2}(\varphi(x) - \varphi(0))\,dx\right|$$

$$\leq \limsup_{\varepsilon \to 0}\int_{|x| \leq \sqrt{\varepsilon}}\underbrace{\frac{|x|}{\varepsilon^2 + x^2}|x|}_{\leq 1}\,\|\varphi\|_{1,0}\,dx = 0.$$

Problem 11.6: We find that

$$\langle D(\log|x|)\,\varphi \rangle = -\langle \log|x|\,\varphi' \rangle = -\int_{\mathbb{R}} \log|x|\,\varphi'(x)\,dx$$

$$= -\lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} \log(-x)\varphi'(x)\,dx + \int_{\varepsilon}^{\infty} \log(x)\varphi'(x)\,dx \right].$$

Now simply perform partial integration in each term separately.

Problem 11.7: First prove that $x \cdot \delta(x) = 0$ and that $x \cdot \mathrm{PV}(1/x) = 1$ (using the regular rules for the product between a polynomial and a Schwartz function). Suppose that $\cdot$ is distributive and can pair any two distributions. Then on the one hand we would have

$$\delta(x) \cdot x \cdot \mathrm{PV}(1/x) = \delta(x) \cdot (x \cdot \mathrm{PV}(1/x)) = \delta(x) \cdot 1 = \delta(x).$$

But we would also have

$$\delta(x) \cdot x \cdot \mathrm{PV}(1/x) = (x \cdot \delta(x)) \cdot \mathrm{PV}(1/x) = 0 \cdot \mathrm{PV}(1/x) = 0.$$

This is a contradiction.

Problem 11.8: Fix $\varphi \in \mathcal{S}$. Set $\alpha = \int \varphi$, and define

$$(3) \qquad \psi(x) = \int_{-\infty}^{x} \big(\varphi(z) - \alpha\,\omega(z) \big)\,dz.$$

Obviously, $\psi \in C^{\infty}$, and

$$(4) \qquad \varphi(x) = \alpha\omega(x) + \psi'(x).$$

Moreover, we find that if $n \geq 1$, then

$$\|\psi\|_{n,k} = \|(1+|x|^2)^{k/2}\psi^{(n)}\|_{\mathrm{u}}$$

$$= \|(1+|x|^2)^{k/2}(\varphi^{(n-1)} - \alpha\omega^{(n-1)})\|_{\mathrm{u}} \leq \|\varphi\|_{n-1,k} + |\alpha|\,\|\omega\|_{n-1,k}.$$

It remains to prove that for any k,

$$\sup_{x}(1+|x|^2)^{k/2}|\psi(x)| < \infty.$$

First consider $x \leq 0$. Then for any k, we have

$$\sup_{x \leq 0}(1+|x|^2)^{k/2}|\psi(x)|$$

$$\leq \limsup_{x \leq 0} \left[(1+|x|^2)^{k/2} \int_{-\infty}^{x} \frac{1}{(1+|y|^{(k+2)/2})}\|\varphi\|_{0,k+2}\,dy \right.$$

$$\left. + |\alpha|(1+|x|^2)^{k/2} \int_{-\infty}^{x} \frac{1}{(1+|y|^{(k+2)/2})}\|\omega\|_{0,k+2}\,dy \right] < \infty.$$

To prove the corresponding estimate for $x \geq 0$, we use that since

$$\underbrace{\int_{-\infty}^{x} \big(\varphi(z) - \alpha\,\omega(z) \big)\,dz}_{=\psi(x)} + \int_{x}^{\infty} \big(\varphi(z) - \alpha\,\omega(z) \big)\,dz = 0,$$

we can also express ψ as

$$\psi(x) = -\int_{x}^{\infty} \big(\varphi(z) - \alpha\,\omega(z) \big)\,dz.$$

Then proceed as in the bound for $x \leq 0$.

Problem 1:

$$\langle D f, \varphi \rangle = -\langle f, \varphi' \rangle = -\int_{-\infty}^{0} (-x)\varphi'(x)\,dx - \int_{0}^{\infty} x\varphi'(x)\,dx$$

$$= \underbrace{[x\varphi(x)]_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0} \varphi(x)\,dx - \underbrace{[x\varphi(x)]_{0}^{\infty}}_{=0} + \int_{-\infty}^{0} \varphi(x)\,dx = \langle g, \varphi \rangle,$$

where

$$g(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

So $D f = g$. (Note that the value of $g(0)$ is irrelevant, any finite value can be assigned.) To compute $D^2 f$, simply differentiate g in the same way. You should find that $D^2 f = 2\delta$.

Problem 2: This is a fairly straight-forward application of the definitions.

Problem 3: Define for $n = 1, 2, 3, \ldots$, the functions

$$\chi_n(x) = \begin{cases} 1 & x \in \left[n - \frac{1}{4^n}, n\right], \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$f(x) = \sum_{n=1}^{\infty} 2^n \chi_n(x).$$

Now prove that both (2) and (3) hold for any k.