Homework set 7 — APPM5450, Spring 2017 — partial solutions

Problem 9.21: Suppose $A \in \mathcal{B}(H)$ is such that
$$\mathrm{Re}(x,\, Ax) \leq 2\alpha \|x\|^2.$$
Prove that the solution $x = x(t)$ of $x'(t) = A\,x(t)$ satisfies
$$\|x(t)\| \leq e^{\alpha t}\,\|x(0)\|.$$

Note: The book may have a typo — the bound seems off by a factor of two. Consider for instance $Ax = 2\alpha x$, then $x(t) = e^{2\alpha t} x(0)$.

Solution: Set $f(t) = \|x(t)\|^2$. Then
$$f'(t) = \frac{d}{dt}(x,\, x) = (x',\, x) + (x,\, x') = (Ax,\, x) + (x,\, Ax) = 2\mathrm{Re}(x,\, Ax) \leq 4\alpha \|x(t)\|^2 = 4\alpha f(t).$$
By the Grönwall inequality, we find
$$\|x(t)\|^2 = f(t) \leq f(0)\,\exp\Big(\int_0^t 4\alpha\, ds\Big) = f(0)\, e^{4\alpha t} = \|x(0)\|^2\, e^{4\alpha t}.$$
Extract the square root to obtain the desired bound.

Problem 9.22: Let A be compact and non-negative. Prove that there exists a unique compact non-negative operator B such that $B^2 = A$.

Solution: Since A is self-adjoint and compact, there is an ON-basis $(\varphi_n)_{n=1}^\infty$ of eigen-vectors of A. $A\,\varphi_n = \lambda_n\,\varphi_n$. We know $|\lambda_n| \to 0$ since A is compact, and $\lambda_n \geq 0$ since A is non-negative.

Existence: Set $B = \sum_{n=1}^\infty \sqrt{\lambda_n}\, P_n$ where $P_n x = (\varphi_n,\, x),\, \varphi_n$. It is easily shown that $B^2 = A$ and that B is compact and non-negative.

Observe that from the construction of B, it follows that if ψ is a vector such that $A\,\psi = \lambda\,\psi$, then $B\,\psi = \sqrt{\lambda}\,\psi$.

Uniqueness: Suppose that C is a non-negative compact operator such that $C^2 = A$. We need to show that $C = B$, where B is the operator constructed above. Since C is compact and self-adjoint, there is an ON-basis $(\psi_n)_{n=1}^\infty$ such that $C\,\psi_n = \mu_n\,\psi_n$. Now observe that
$$A\,\psi_n = C^2\,\psi_n = C\,(\mu_n\,\psi_n) = \mu_n^2 \psi_n$$
so ψ_n is an eigenvector of A with eigenvalue μ_n^2. It follows that $B\,\psi_n = \sqrt{\mu_n^2}\,\psi_n = \mu_n\,\psi_n = C\,\psi_n$. (We know that $\sqrt{\mu_n^2} = \mu_n$ since C must be non-negative, which implies that $\mu_n \geq 0$.)

Problem 1: Consider the Hilbert space $H = \mathbb{C}^n$. Let $A \in \mathcal{B}(H)$, let $(e^{(j)})_{j=1}^n$ be the canonical basis, and let A have the representation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

in the canonical basis. We define the *Hilbert-Schmidt norm* of A as

$$\|A\|_{\text{HS}} = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

(a) Let $(\varphi^{(j)})_{j=1}^n$ be any ON-basis for H. Show that $\|A\|_{\text{HS}}^2 = \sum_{j=1}^n \|A\varphi^{(j)}\|^2$.

(b) Show that $\|A\| \leq \|A\|_{\text{HS}} \leq \sqrt{n}\|A\|$ for any $A \in \mathcal{B}(H)$.

(c) Find $G, H \in \mathcal{B}(H)$ such that $\|G\|_{\text{HS}} = \|G\|$ and $\|H\|_{\text{HS}} = \sqrt{n}\|H\|$.

Solution:

(a) Let $r^{(i)}$ denote the i'th row of A. Then

$$\sum_{j=1}^n \|A\varphi^{(j)}\|^2 = \sum_{j=1}^n \sum_{i=1}^n \|(r^{(i)},\, \varphi^{(j)})\|^2 = \{\text{Parseval}\} = \sum_{i=1}^n \|r^{(i)}\|^2 = \|A\|_{\text{HS}}^2.$$

(b) For any x a simple application of Cauchy-Schwartz yields

$$\|Ax\|^2 = \sum_{i=1}^n \|(r^{(i)},\, x)\|^2 \leq \sum_{i=1}^n \|r^{(i)}\|^2\, \|x\|^2 = \|A\|_{\text{HS}}^2 \|x\|^2.$$

It follows that $\|A\| \leq \|A\|_{\text{HS}}$. Next, let i be such that $\|r^{(i)}\| = \max_j \|r^{(j)}\|$. Then

$$\|A\|_{\text{HS}}^2 = \sum_{j=1}^n \|r^{(j)}\|^2 \leq n\, \|r^{(i)}\|^2 = n\, \|A^*\, e_i\|^2 \leq n\, \|A^*\| = n\, \|A\|,$$

where e_i denotes the i'th canonical basis vector.

(c) For instance, let G be the matrix consisting of all ones. Then, the singular value decomposition of G is $G = \sqrt{n}\, g\, g^*$, where $g = (1, 1, 1, \dots, 1)/\sqrt{n}$. Consequently, $\|G\| = \sqrt{n}$. It is a trivial computation to shot that $\|G\|_{\text{HS}} = \sqrt{n}$.

Next, let H be the identity matrix. Then obviously $\|H\| = 1$ since $\|Hx\| = \|x\|$ for any vector x. But $\|H\|_{\text{HS}} = \sqrt{n}$.

Problem 2: Let H be a separable Hilbert space, and let $A \in \mathcal{B}(H)$. Suppose that H has an ON-basis $(\varphi^{(j)})_{j=1}^{\infty}$ such that

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 < \infty.$$

Prove that if $(\psi^{(j)})_{j=1}^{\infty}$ is any other ON-basis, then

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{\infty} \|A\psi^{(j)}\|^2.$$

Solution: Set

$$\alpha_{ji} = (A\,\varphi^{(j)},\, \psi^{(i)}) = (\varphi^{(j)},\, A^*\,\psi^{(i)})$$

and

$$\beta_{ik} = (A^*\,\psi^{(i)},\, \psi^{(k)}) = (\psi^{(i)},\, A\,\psi^{(k)}).$$

The proof consists of four applications of Parseval:

$$\sum_{j=1}^{\infty} \|A\,\varphi^{(j)}\|^2 = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |\alpha_{ji}|^2 = \sum_{i=1}^{\infty} \|A^*\,\psi^{(i)}\|^2 = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |\beta_{ik}|^2 = \sum_{k=1}^{\infty} \|A\,\psi^{(k)}\|^2.$$

Note that the interchanges of summation order are permissible as all terms are non-negative.

Problem 3: Consider the linear space $L = \mathbb{R}^2$. Define for $x = (x_1,\, x_2) \in L$ the seminorms

$$p_1(x) = |x_1|, \qquad p_2(x) = |x_2|.$$

Construct for $x \in L$, $j \in \{1,\, 2\}$, and $\varepsilon \in (0, \infty)$, the sets

$$\mathcal{B}_{x,j,\varepsilon} = \{y \in L :\, p_j j(x - y) < \varepsilon\}.$$

Describe these sets geometrically. What is the topology generated by the collection of semi-norms $\{p_1\}$? Is it Hausdorff? What is the topology generated by the collection of semi-norms $\{p_1,\, p_2\}$? Is it Hausdorff?

Solution:

For $x = (x_1,\, x_2)$, the set $\mathcal{B}_{x,1,\varepsilon}$ is a vertical strip of width 2ε centered around x_1. The set $\mathcal{B}_{x,2,\varepsilon}$ is a horizontal strip of width 2ε centered around x_2.

The topology $\mathcal{T}_1$ generated by $\{p_1\}$ is the topology on the real line. In other words, $\Omega \in \mathcal{T}_1$ iff $\Omega = \Omega_1 \times \mathbb{R}$ where Ω_1 is an open set on the line. This topology is not Hausdorff. For a counter-example, set $x = (0,\, 0)$ and $y = (0,\, 1)$. Then if $\Omega \in \mathcal{T}_1$ we have

$$x \in \Omega \qquad \Leftrightarrow \qquad y \in \Omega.$$

As far as $\mathcal{T}_1$ is concerned, the points x and y are not distinct.

The topology generated by $\{p_1,\, p_2\}$ has as its base $\mathcal{B}$ intersections of open sets in $\mathcal{T}_1$ and $\mathcal{T}_2$. This means that $\mathcal{B}$ consists of all open rectangles in the plane. These generate the standard topology on $\mathbb{R}^2$, which is Hausdorff.