Homework set 6 — APPM5450, Spring 2017

If you didn't complete all of the problems 9.1 – 9.11 last week, then continue working on that.

Problem 1: Let H_1 and H_2 be Hilbert spaces, let $U : H_1 \to H_2$ be unitary, and let $A \in \mathcal{B}(H_1)$. Define $\tilde{A} \in \mathcal{B}(H_2)$ by $\tilde{A} = U\,A\,U^{-1}$. Prove that

- $\sigma_{\rm p}(A) = \sigma_{\rm p}(\tilde{A})$
- $\sigma_{\rm c}(A) = \sigma_{\rm c}(\tilde{A})$
- $\sigma_{\rm r}(A) = \sigma_{\rm r}(\tilde{A})$

Problem 2: Let A be a self-adjoint compact operator. For $\lambda \in \rho(A)$, set $R_\lambda = (A - \lambda I)^{-1}$ as usual. Construct the spectral decomposition of R_λ. Use it to prove that

$$||R_\lambda|| = \frac{1}{\mathrm{dist}(\lambda,\, \sigma(A))} = \frac{1}{\inf_{\mu \in \sigma(A)} |\lambda - \mu|}.$$

Problem 3: Consider the Hilbert space $H = L^2(I)$, where $I = [-\pi,\, \pi]$. Define

$$\Omega_t = \{u \in H :\; u(x) = 0\ \forall\, x \geq t\}.$$

Note that Ω_t is a closed linear subspace of H. Define $P(t)$ as the orthogonal projection onto Ω_t. Consider the self-adjoint operator $A \in \mathcal{B}(H)$ defined by

$$[Au](x) = x\,u(x).$$

(a) Prove that Ω_t is an invariant subspace of A for every $t \in \mathbb{R}$.

(b) Prove that if $a < b \leq c < d$, then $\mathrm{ran}(P(b) - P(a)) \perp \mathrm{ran}(P(d) - P(c))$. Conclude that for any numbers $-\pi = t_0 < t_1 < t_2 < \cdots < t_n = \pi$, it is the case that

$$H = \mathrm{ran}[P(t_1) - P(t_0)] \oplus \mathrm{ran}[P(t_2) - P(t_1)] \oplus \cdots \oplus \mathrm{ran}[P(t_n) - P(t_{n-1})],$$

where each term is an invariant subspace of A.

(c) For a positive integer n, set $h = 2\pi/n$, and $\lambda_j = -\pi + h\,j$. Define the operator

$$A_n = \sum_{j=1}^{n} \lambda_j \left(P(\lambda_j) - P(\lambda_{j-1})\right). \tag{1}$$

Prove that $||A - A_n|| \leq 2\pi/n$. Conclude that $A_n \to A$ in norm.

Note: There is a spectral theorem for all self-adjoint operators on Hilbert spaces (all normal ones, even). For operators with continuum spectra such as A, the spectral decomposition of A involves so called "projection valued measures". The sum (1) is a Riemann-Stieltjes sum of the integral

$$A = \int_{-\pi}^{\pi} \lambda\, dP(\lambda).$$