Homework set 4 — APPM5450 — Spring 2017

From the textbook: 8.15.

Problem 1: Consider the Hilbert space $H = l^2(\mathbb{N})$; let e_n denote the canonical basis vectors. Which of the following sequences converge weakly? Which have convergent subsequences?

(a) $x_n = n\, e_n$.

(b) $y_n = n^{-1/2} \sum_{j=1}^{n} e_j$.

(c) $x_n = e_n + e_m$ where $m = 1 + \text{mod}(n, 2)$.

Problem 2: Consider the Hilbert space $H = L^2([-\pi,\, \pi])$, and the sequence of functions $\varphi_n(x) = x^2\, \sin(n\, x)$. Does $(\varphi_n)_{n=1}^{\infty}$ converge strongly in H? Does $(\varphi_n)_{n=1}^{\infty}$ converge weakly in H? If you answer yes to either question, specify the limit.

Problem 3: Let A denote a self-adjoint operator on a Hilbert space H. Let u denote an element of H and set $u_n = e^{i\,n\,A}\, u$. Prove that $(u_n)_{n=1}^{\infty}$ has a weakly convergent subsequence.

Problem 4: Let H_1 and H_2 be two Hilbert spaces. Let $U : H_1 \rightarrow H_2$ be a unitary operator, and let $A_1 \in \mathcal{B}(H_1)$ be a self-adjoint operator. Define the operator $A_2 \in \mathcal{B}(H_2)$ by $A_2 = U\, A_1\, U^{-1}$. Prove that A_2 is self-adjoint.

Problem 5 (optional): Consider the Hilbert space $H = L^2([-\pi,\, \pi])$, and let $\mathcal{P}$ denote the set of trigonometric polynomials (which is dense in H. For $u \in \mathcal{P}$, let A denote the operator $Au = 100\, u + 18\, u'' + u''''$. Prove that

$$\sup_{u \in \mathcal{P},\ ||u||=1} \langle Au,\, u\rangle = \infty.$$

Conclude that A cannot be extended to a bounded linear operator on H. Prove that for $u, v \in \mathcal{P}$, $\langle A\, u,\, v\rangle = \langle u,\, A\, v\rangle$. Determine

$$\inf_{u \in \mathcal{P},\ ||u||=1} \langle Au,\, u\rangle.$$

Prove that

$$\langle u,\, v\rangle_A = \langle A\, u,\, v\rangle$$

is a bilinear form on $\mathcal{P}$. Prove that on $\mathcal{P}$, the norm $||\cdot||_A$ induced by $\langle \cdot,\, \cdot\rangle_A$ is equivalent to the norm

$$||u||_{H^2(\mathbb{T})} = \sqrt{||u||^2_{L^2(\mathbb{T})} + ||u''||^2_{L^2(\mathbb{T})}}.$$

Conclude that the closure of $\mathcal{P}$ under the norm $||\cdot||_A$ is the space $H^2(\mathbb{T})$ (as defined in Section 7.2).