Solutions for Homework 1 — APPM5450 — Spring 2017

Problem 7.1:

(a) Fix $\delta > 0$. For $x \in [-\delta/2,\, \delta/2]$ we have $1 + \cos x \geq 1 + \cos \frac{\delta}{2}$ so

$$\text{(1)} \qquad \frac{1}{c_n} = \int_{\mathbb{T}} (1+\cos x)^n \, dx \geq \int_{-\delta/2}^{\delta/2} \left(1 + \cos\frac{\delta}{2}\right)^n \, dx = \delta \left(1 + \cos\frac{\delta}{2}\right)^n.$$

Analogously, we find that

$$\text{(2)} \qquad \int_{|x|\geq\delta} c_n (1+\cos x)^n \, dx \leq \int_{|x|\geq\delta} c_n (1+\cos\delta)^n \, dx \leq c_n 2\pi (1+\cos\delta)^n.$$

Inserting (1) into (2) and taking the limit, we find (since $1 + \cos\delta < 1 + \cos(\delta/2)$)

$$\lim_{n\to\infty} \int_{|x|\geq\delta} c_n (1+\cos x)^n dx \leq \limsup_{n\to\infty} \frac{2\pi}{\delta} \left(\frac{1+\cos\delta}{1+\cos(\delta/2)}\right)^n = 0.$$

(b) See lecture notes.

(c) No, since any function in $\mathcal{P}$ is periodic. Consider for instance $f(x) = x$. Then for any $g \in \mathcal{P}$

$$||f-g||_{\mathrm{u}} \geq \max(|f(0)-g(0)|,\, |f(2\pi)-g(2\pi)|) = \max(|g(0)|,\, |2\pi - g(0)|) \geq \pi.$$

Problem 7.2: With $e_n(x) = e^{inx}/\sqrt{2\pi}$ we set

$$f_N(x) = \sum_{n=-N}^{N} \alpha_n \, e_n(x), \qquad \alpha_n = (e_n, f).$$

Set $\beta = 1/\sqrt{2\pi}$. Then

$$f_N(x) = \sum_{n=-N}^{N} \int_{-\pi}^{\pi} \beta e^{-iny} f(y) dy \, \beta e^{inx} = \int_{-\pi}^{\pi} \underbrace{\beta^2 \sum_{n=-N}^{N} e^{in(x-y)}}_{=: D_N(x-y)} f(y) dy.$$

We will next simplify the kernel D_N. To this end, set $\alpha = e^{ix}$. Then

$$D_N = \beta^2 \sum_{n=-N}^{N} \alpha^n.$$

Moreover,

$$\alpha \, D_N = \beta^2 \sum_{n=-N}^{N} \alpha^{n+1}.$$

In consequence,

$$(1-\alpha)\, D_N = \beta^2 (\alpha^{-N} - \alpha^{N+1}).$$

It follows that

$$D_N = \beta^2 \frac{\alpha^{-N} - \alpha^{N+1}}{1-\alpha} = \beta^2 \frac{\alpha^{-(N+1/2)} - \alpha^{N+1/2}}{\alpha^{-1/2} - \alpha^{1/2}} = \frac{1}{2\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

This proves part (a).

Next we set

$$g_N = \frac{1}{N+1}\sum_{n=0}^{N} f_0 = \frac{1}{N+1}\sum_{n=0}^{N} D_n * f = \underbrace{\left(\frac{1}{N+1}\sum_{n=0}^{N} D_n\right)}_{=:F_N} * f.$$

It remains to simplify F_N. We have

$$F_N = \frac{1}{N+1}\sum_{n=0}^{N} D_n = \frac{1}{N+1}\sum_{n=0}^{N} \beta^2 \frac{\alpha^{-n} - \alpha^{n+1}}{1-\alpha} = \frac{\beta^2}{(N+1)(1-\alpha)}\left[\frac{(1/\alpha)^{N+1} - 1}{1/\alpha - 1} - \frac{\alpha^{N+2} - \alpha}{\alpha - 1}\right]$$
$$= \frac{\beta^2\alpha}{(N+1)(1-\alpha)^2}\left[\alpha^{-(N+1)} - 2 + \alpha^{N+1}\right] = \frac{\beta^2\alpha}{(N+1)(\alpha^{-1/2} - \alpha^{1/2})^2}\left[\alpha^{-(N+1)/2} - \alpha^{(N+1)}\right]^2$$
$$= \frac{\beta^2\alpha}{(N+1)(-2i\sin(x/2))^2}\left[-2i\sin\frac{(N+1)x}{2}\right]^2 = \frac{\beta^2\alpha}{(N+1)(\sin(x/2))^2}\left[\sin\frac{(N+1)x}{2}\right]^2.$$

This proves part (b).

For (c), we observe that D_N takes on non-negative values, so it is not an approximate identity. Convolution by D_N provides the best approximation in the L^2-norm, but it does not guarantee convergence in the uniform norm. In contrast, convolution by F_N does provide convergence in the uniform norm as long as $f \in C(\mathbb{T})$.

Problem 7.3: Start by proving that the two putative bases are in fact orthonormal sets. Then it remains to prove that their closures span the set.

Fix an $f \in L^2(J)$ with $J = [0,\pi]$. To construct a sequence f_N such that $||f - f_N||_{L^2(J)} \to 0$, extend f to the function

$$\bar{f}(x) = \begin{cases} f(x) & x \geq 0, \\ -f(-x) & x < 0. \end{cases}$$

Then let f_N be the standard Fourier series of $\bar{f}$. Prove that the terms in this series are all sine functions. Since the exponentials form a basis, we know that $||\bar{f} - f_N||_{L^2(I)} \to 0$ where $I = [-\pi, \pi]$. Since $||\bar{f} - f_N||_{L^2(I)} = \sqrt{2}||f - f_N||_{L^2(J)}$, we then find that $f_N \to f$ in $L^2(J)$.

To prove that the cosines form a basis, repeat the argument, but do it with the symmetric continuation of f instead of the anti-symmetric one. In other words, set

$$\tilde{f}(x) = \begin{cases} f(x) & x \geq 0, \\ f(-x) & x < 0, \end{cases}$$

and then use that the Fourier series for $\tilde{f}$ involves only cosines.

Problem 7.4: This is a straight-forward calculation. You may want to look the correct answer up in a table to make sure you got the answer right.

Problem 7.5: The argument for the case $d = 1$ was done in class (see posted lecture notes). This argument can easily be modified to the case of d dimensions. Let f_N denote the partial Fourier sum. We need to prove that (f_N) is Cauchy with respect to the uniform norm. If $M < N$, we find

$$
\begin{aligned}
|f_M(x) - f_N(x)| = \left| \sum_{M<|n|\leq N} \alpha_n e_n(x) \right| &\leq \sum_{M<|n|\leq N} |\alpha_n| \\
&\leq \left(\sum_{M<|n|\leq N} |n|^{-2k} \right)^{1/2} \left(\sum_{M<|n|\leq N} |n|^{2k} |\alpha_n|^2 \right)^{1/2} \\
\sim \left(\int_{M\leq|x|\leq N} |x|^{-2k} dx \right)^{1/2} ||f||_{H^k} &\sim \left(\int_M^N r^{-2k} r^{d-1} dr \right)^{1/2} ||f||_{H^k} \\
&\leq \left(\int_M^\infty r^{-2k} r^{d-1} dr \right)^{1/2} ||f||_{H^k} = \frac{1}{\sqrt{2k-d}\, M^{k-d/2}} ||f||_{H^k}.
\end{aligned}
$$

Problem 1: Suppose that H is a Hilbert space, and that $(\psi_n)_{n=1}^{\infty}$ is an ON-set in H. Let $\mathcal{P}$ denote the set of finite linear combinations of elements in $(\psi_n)_{n=1}^{\infty}$. Prove that:

$$(\psi_n)_{n=1}^{\infty} \text{ is a basis for } H \quad \Leftrightarrow \quad \mathcal{P} \text{ is dense in } H.$$

Solution: Suppose first that $(\psi_n)_{n=1}^{\infty}$ is a basis. Given any $f \in H$, define its partial expansion in (ψ_n) as usual:

$$f_N = \sum_{n=1}^{N} (\psi_n,\, f)\, \psi_n \tag{3}$$

Since (ψ_n) is a basis, we know that $f_N \to f$ in norm. Since $f_N \in \mathcal{P}$, this proves that any function can be approximated arbitrarily well be functions in $\mathcal{P}$.

Suppose next that $\mathcal{P}$ is dense. Fix an $f \in H$, and define its partial expansion f_N as in (3). We need to prove that $f_N \to f$. Fix any $\varepsilon > 0$. Since $\mathcal{P}$ is dense, there is a $g \in \mathcal{P}$ such that $||f - g|| < \varepsilon$. Let N be a number such that $g \in \text{Span}(\psi_1,\, \psi_2,\, \dots,\, \psi_N) =: \mathcal{P}_N$. Now suppose that that $M \geq N$. Then since $g \in \mathcal{P}_M$, and f_M is the best possible approximant within $\mathcal{P}_M$, we find

$$||f - f_M|| \leq ||f - g|| < \varepsilon.$$

This shows that $f_N \to f$.

Problem 2: Suppose that $f, g \in C(\mathbb{T})$. Prove that:

(a) $f * g \in C(\mathbb{T})$.

(b) $f * g = g * f$.

Solution:

(a) Set $h = f * g$. That h is periodic follows directly from the periodicity of f:

$$h(x + 2\pi) = \int_{\mathbb{T}} f(x + 2\pi - y) g(y) dy = \int_{\mathbb{T}} f(x - y) g(y) dy = h(x).$$

Next we prove continuity. Fix $\varepsilon > 0$. Since f is uniformly continuous, there is a $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon/(2\pi||g||)$ whenever $|x - x'| < \delta$. Now suppose that $|x - x'| < \delta$. Then

$$|h(x) - h(x')| = |\int_{\mathbb{T}} (f(x-y) - f(x'-y)) g(y) dy| \leq \int_{\mathbb{T}} |f(x-y) - f(x'-y)|\, |g(y)| dy \leq \int_{\mathbb{T}} \frac{\varepsilon}{2\pi||g||}\, ||g|| dy = \varepsilon.$$

(b) Simply use the change of variables $z = x - y$ in the integral. You need to verify that the limits and the minus signs work out as they should, but that should not be hard.