THE FOURIER TRANSFORM ON $\mathcal{S}^*(\mathbb{R}^d)$

Suppose first that $T \in \mathcal{S}^*$ is a smooth & compactly supported function, $T \in C_c^\infty(\mathbb{R}^d)$. Then

$$\langle \hat{T}, \varphi \rangle = \int_{\mathbb{R}^d} \underbrace{\rho^d \int_{\mathbb{R}^d} e^{-ix\cdot t}\, T(x)\,dx}_{=\hat{T}(x)}\, \varphi(t)\,dt \;\overset{*}{=}$$

$\boxed{\text{The integrand is absolutely summable, so we can invoke Fubini to interchange the integration order}}$

$$= \int_{\mathbb{R}^d} T(x) \underbrace{\rho^d \int e^{-ix\cdot t}\, \varphi(t)\,dt}_{=\hat{\varphi}(t)}\, dx = \langle T, \hat{\varphi} \rangle$$

Since we've already proven that $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is continuous, we can now trivially define $\mathcal{F}: \mathcal{S}^* \to \mathcal{S}^*$ by duality.

<u>Def^n</u> For $T \in \mathcal{S}^*(\mathbb{R}^d)$, define $\hat{T} = \mathcal{F}T$ by

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle.$$

Similarly, define $\check{T} = \mathcal{F}^* T$ by $\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle$.

It follows immediately that

We proved earlier that if $\varphi \in \mathcal{S}$, then $\mathcal{F}\mathcal{F}^*\varphi = \varphi$.

$$\langle \mathcal{F}^*\mathcal{F}T, \varphi \rangle = \langle \mathcal{F}T, \mathcal{F}^*\varphi \rangle = \langle T, \overbrace{\mathcal{F}\mathcal{F}^*\varphi} \rangle = \langle T, \varphi \rangle$$

So $\mathcal{F}^*$ is the inverse of $\mathcal{F}$ on $\mathcal{S}^*$. To sum up:

<u>Prop^n</u> The map $\mathcal{F}: \mathcal{S}^* \to \mathcal{S}^*$ is continuous, bijective, and ~~each~~ its inverse is continuous as well.

<u>Example</u> $T = \delta$ ← Dirac delta function

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle = \hat{\varphi}(0) = \rho^d \int \cancel{\varphi}\, e^{-i0\cdot x} \varphi(x)\,dx = \rho^d \int \varphi(x)\,dx = \langle \rho^d, \varphi \rangle$$

So $\hat{\delta} = \dfrac{1}{(2\pi)^{d/2}}$

The Fourier transform on $L^2(\mathbb{R}^d)$

For now, we switch to complex-valued functions on $\mathcal{S}$ & $\mathcal{S}^*$. Nothing that we've done changes.

Def^n Let $(\cdot,\cdot)$ denote the inner product, $(u,v) = \int \overline{u(x)}\, v(x)\, dx$. Define $L^2(\mathbb{R}^d)$ by taking the closure of $\mathcal{S}$ w.r.t. the $\|\cdot\|_2^{\mathbb{R}^d}$ norm.

Lemma If $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, then $(\varphi,\psi) = (\hat\varphi, \hat\psi)$.

Proof $(\varphi,\psi) = \int \bar\varphi\, \psi = \langle \bar\varphi, \psi \rangle = \langle \bar\varphi, \check{\hat\psi} \rangle = \langle \check{\bar\varphi}, \hat\psi \rangle$

We have $\check{\bar\varphi}(t) = \beta^d \int e^{ixt} \bar\varphi(x)\,dx = \overline{\beta^d \int e^{-ixt} \varphi(x)\, dx} = \overline{\hat\varphi(t)}$ so

$$(\varphi,\psi) = \langle \check{\bar\varphi}, \hat\psi \rangle = \langle \bar{\hat\varphi}, \hat\psi \rangle = (\hat\varphi, \hat\psi).$$

As a consequence of the lemma: $\|\mathcal{F}\varphi\|_2^2 = (\hat\varphi, \hat\varphi) = (\varphi,\varphi) = \|\varphi\|_2^2$

Thus $\mathcal{F} : \mathcal{S} \to L^2(\mathbb{R}^d)$ is an isometric map.

Since $\mathcal{S}$ is dense in $L^2(\mathbb{R}^d)$, we can uniquely extend $\mathcal{F}$ to all of $L^2(\mathbb{R}^d)$.

Def^n We define $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ as follows:

For $f \in L^2(\mathbb{R}^d)$, pick $\varphi_n \in \mathcal{S}$ s.t. $\|f - \varphi_n\|_2 \to 0$. Set $\overset{=\mathcal{F}f}{\hat f} = \lim_{n\to\infty} \hat\varphi_n$ ← Limit is in L^2

Define $\mathcal{F}^* : L^2 \to L^2$ analogously, $\check f = \lim_{n\to\infty} \check\varphi_n$.

It follows immediately that $\mathcal{F}^* = \mathcal{F}^{-1}$ and thus that $\mathcal{F}$ is bijective.

To summarize:

Thm The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is

* Bijective (one-to-one & onto)
* Preserves the norm: $\int |u(x)|^2 = \int |\hat{u}(x)|^2$
* Preserves the inner product: $\int \overline{u(x)} v(x) = \int \overline{\hat{u}(t)}\, \hat{v}(t)\, dt$
* Is continously invertible.

In short, it is a "unitary map" or a "Hilbert space isomorphism".

Note We defined $\hat{f}$ as the limit $\hat{f} = \lim \hat{\varphi}_n$ where $\varphi_n \in \mathcal{S}$.
In practise, it is enough to pick $\varphi_n \in L^2 \cap L^1$, for instance

$$\hat{f}(t) = \lim_{R\to\infty} \beta^d \int_{|x|\le R} e^{-ix\cdot t} f(x)\,dx = \lim_{\varepsilon\to 0} \beta^d \int e^{-ix\cdot t} e^{-\varepsilon|x|^2} f(x)\,dx$$

What is the spectrum of $\mathcal{F}$?

First we note that since $\mathcal{F}$ is unitary, we must have $\sigma(\mathcal{F}) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Moreover, note that ~~since~~ $\mathcal{F}^2 = \mathcal{F}\mathcal{F} = R\mathcal{F}^*\mathcal{F} = R$, and so $\mathcal{F}^4 = R^2 = I$.

This ~~...~~ implies that if $\lambda \in \sigma(\mathcal{F})$, then $\lambda^4 = 1$ and so $\lambda \in \{1, -1, i, -i\}$.

(Formally: $\mathcal{F}^4 = \left(\int_{\sigma(\mathcal{F})} \lambda\, dP(\lambda)\right)^4 = \int_{\sigma(\mathcal{F})} \lambda^4\, dP(\lambda) = I \Rightarrow \lambda^4 = 1$)

In fact the four numbers $1, -1, i, -i$ are all evals of infinity multiplicity and we have explicit formulas for the eigenvectors!

In one dimension, ~~these~~ these are the so call Hermite functions $(\varphi_n)_{n=0}^{\infty}$, which form an ON-basis for $L^2(\mathbb{R})$

$$\varphi_n(x) = \alpha_n e^{x^2/2} \left(\frac{d}{dx}\right)^n e^{-x^2} = \alpha_n H_n(x) e^{-x^2/2}$$ where H_n are the Hermite polynomials

(The H_n are constructed for instance by applying Gram-Schmidt to $\{1, x, x^2, \ldots\}$ w.r.t. $(u,v) = \int \bar{u} v e^{-x^2} dx$.)

$$\alpha_n = \frac{1}{\sqrt{n!\, 2^n \sqrt{\pi}}}$$

It turns out that $\mathcal{F}\varphi_n = (-i)^n \varphi_n$ and so

$$L^2(\mathbb{R}^d) = \ker(\mathcal{F}-I) \oplus \ker(\mathcal{F}+iI) \oplus \ker(\mathcal{F}+I) \oplus \ker(\mathcal{F}-iI) =$$

$$= \mathrm{span}(\varphi_{4n}) \oplus \mathrm{span}(\varphi_{4n+1}) \oplus \mathrm{span}(\varphi_{4n+2}) \oplus \mathrm{span}(\varphi_{4n+3})$$

With this knowledge, we could also have defined $\mathcal{F}$ via

$$\mathcal{F}\Big[\sum_{n=0}^{\infty} \alpha_n \varphi_n\Big] = \sum_{n=0}^{\infty} \alpha_n (-i)^n \varphi_n .$$

Sobolev Spaces

When is $\partial^\alpha f$ an L^2 function?

$$\partial^\alpha f \in L^2 \iff \int |\partial^\alpha f|^2 < \infty \iff \int |(-it)^\alpha \hat{f}(t)|^2 < \infty \iff (-it)^\alpha \hat{f} \in L^2$$

Let s be a non-negative integer and define the Sobolev space $H^s(\mathbb{R}^d)$ by

$$H^s(\mathbb{R}^d) = \{f : \partial^\alpha f \in L^2 \ \forall \alpha : |\alpha| \le s\} =$$

$$= \{f : (-it)^\alpha \hat{f} \in L^2 \ \forall \alpha : |\alpha| \le s\} =$$

$$= \{f : (1+|t|^2)^{s/2} \hat{f} \in L^2\} = \{f : \int (1+|t|^2)^s |\hat{f}(t)|^2 dt < \infty\}.$$

The definition is readily extended to any $s \in \mathbb{R}$, not just positive integers.

Lemma (Sobolev Embedding) If $f \in H^s(\mathbb{R}^d)$, for some $s > d/2$, then $f \in C_0(\mathbb{R}^d)$.
More generally, if $f \in H^s(\mathbb{R}^d)$ for some $s > \frac{d}{2} + k$, then $f \in C_0^k(\mathbb{R}^d)$.