APPM5450 — Applied Analysis: Section exam 2
8:30 – 9:50, March 19, 2014. Closed books.

Problem 1: (12p) Let A be a self-adjoint bounded compact linear operator on a separable Hilbert space H. Which statements are necessarily true (no motivation required):

(a) H has an ON-basis of eigenvectors of A.
(b) If $(e_n)_{n=1}^{\infty}$ is an ON-sequence, then $\lim_{n\to\infty} ||A\,e_n|| = 0$.
(c) For any $\lambda \in \mathbb{C}$, the subspace $\ker(A - \lambda I)$ is necessarily finite dimensional.
(d) $\sigma_{\rm c}(A) = \emptyset$.
(e) $\sigma_{\rm r}(A) = \emptyset$.
(f) $||A||$ is necessarily an eigenvalue of A.

Problem 2: (12p) Let P be a projection on a Hilbert space H. Which of the following statements are necessarily correct (no motivation required):
(a) The spectral radius $r(P)$ is either precisely zero or precisely one.
(b) $\sigma(P) \subseteq \{\lambda \in \mathbb{C} :\ |\lambda| \leq 1\}$.
(c) $\sigma(P) \subseteq \mathbb{R}$.
(d) If P is orthogonal, then $\sigma(P) \subseteq \{0, 1\}$.
(e) If $||Px|| = ||x||$ for every $x \in H$, then P is necessarily the identity.
(f) If there exist $x \in \text{ran}(P)$ and $y \in \ker(P)$ such that $\langle x,\, y\rangle \neq 0$, then $||P|| > 1$.

Problem 3: (25p) Let H be a Hilbert space, and let A be a bounded linear operator on H, so that $A \in \mathcal{B}(H)$.

(a) Define the *resolvent set* $\rho(A)$.
(b) Prove that $\rho(A)$ is an open set.

Problem 4: (25p) Define a map $T :\ \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ via

$$T(\varphi) = \lim_{\varepsilon \searrow 0}\left(\int_{-\infty}^{-\varepsilon} \frac{1}{x}\varphi(x)\,dx + \int_{\varepsilon}^{\infty} \frac{1}{x}\varphi(x)\,dx\right).$$

Prove that T is a continuous functional on $\mathcal{S}$. (You do not need to prove linearity.) What can you say about the order of T?

Note: Recall that the *order* of a distribution is the lowest number m for which a bound of the form $|T(\varphi)| \leq C \sum_{\ell \leq k} \sum_{|\alpha| \leq m} ||\varphi||_{\ell,\alpha}$ holds.

Problem 5: (24p) Consider the Hilbert space $H = L^2(\mathbb{R})$. For this problem, we define H as the closure of the set of all compactly supported smooth functions on $\mathbb{R}$ under the norm

$$||u|| = \left(\int_{-\infty}^{\infty} |u(x)|^2\,dx\right)^{1/2}.$$

Which of the following sequences converge weakly in H? Motive your answers briefly.

(a) $(u_n)_{n=1}^{\infty}$ where $u_n(x) = \begin{cases} 1 - |x - n|, & \text{for } x \in [n-1,\, n+1], \\ 0, & \text{for } x \in (-\infty, n-1) \cup (n+1, \infty). \end{cases}$

(b) $(v_n)_{n=1}^{\infty}$ where $v_n(x) = \sin(n\,x)\,e^{-x^2}$.

(c) $(w_n)_{n=1}^{\infty}$ where $w_n(x) = \begin{cases} 1 - |x/n - 1| & \text{for } x \in [0, 2n] \\ 0 & \text{for } x \in (-\infty, 0) \cup (2n, \infty). \end{cases}$