APPM5450 — Applied Analysis: Section exam 1 — Solutions 8:30 – 9:50, February 10, 2014. Closed books.

Problem 1: Consider the Hilbert space $H = \ell^2(\mathbb{N})$, and the operator

$$A(x_1,\, x_2,\, x_3,\, \dots) = (\lambda_1 x_1,\, \lambda_2 x_2,\, \lambda_3 x_3,\, \dots),$$

where $(\lambda_n)_{n=1}^{\infty}$ is a bounded sequence of complex numbers.

(a) Prove that $||A|| = \sup_n |\lambda_n|$.

(b) Give minimal conditions on the numbers λ_n that ensure that A is, respectively:
  (i) self-adjoint,
  (ii) non-negative,
  (iii) positive,
  (iv) coercive.
  Motivate your claims.

Solution:

(a) Set $M = \sup_n |\lambda_n|$. Then

$$||Ax||^2 = \sum_{n=1}^{\infty} |\lambda_n x_n|^2 \leq \sum_{n=1}^{\infty} M^2 |x_n|^2 = M^2 ||x||^2.$$

Conversely, let e_n denote the n'th canonical unit vector. Then

$$||A|| = \sup_{||x||=1} ||Ax|| \geq ||Ae_n|| = |\lambda_n|.$$

Take the supremum to get $||A|| \geq \sup_n |\lambda_n| = M$.

(b) We find

$$\langle Ax, y\rangle = \sum_n \overline{\lambda_n x_n}\, y_n = \sum_n \overline{x_n}\, \overline{\lambda_n} y_n = \langle x, A^* y\rangle,$$

where

$$A^*(x_1,\, x_2,\, x_3,\, \dots) = (\bar{\lambda}_1 x_1,\, \bar{\lambda}_2 x_2,\, \bar{\lambda}_3 x_3,\, \dots),$$

It follows that A is self-adjoint iff every λ_j is purely real.

Next suppose that A is S-A, then $\langle Ax, x\rangle = \sum_{n=1}^{\infty} |x_n|^2$.

It follows immediately that A is non-negative iff $\lambda_n \geq 0$ for every n, and that A is positive iff $\lambda_n > 0$ for every n.

Set $c = \inf_n \lambda_n$. If $c \leq 0$, then $\inf_{||x||=1} \langle Ax, x\rangle \leq \inf_n \langle Ae_n, e_n\rangle = \inf_n |\lambda_n| \leq 0$, so in this case, A is not coercive. Conversely, if $c > 0$, then $\langle Ax, x\rangle = \sum_n \lambda_n |x_n|^2 \geq c \sum_n |x_n|^2 = c||x||^2$ so A is coercive.

Problem 2: Let $\mathbb{T}$ denote the unit circle parameterized using the interval $I = [-\pi, \pi]$ as usual, and define the function $f \in L^2(\mathbb{T})$ via

$$f(x) = \begin{cases} 1 & \text{when } |x| \le \pi/2, \\ 0 & \text{when } |x| > \pi/2. \end{cases}$$

(a) Compute the Fourier series of f.
(b) Determine for which $s \in \mathbb{R}$ it is the case that f belongs to the Sobolev space $H^s(\mathbb{T})$.
(c) Now define a function $g \in L^2(\mathbb{T}^2)$ via

$$g(x_1, x_2) = f(x_1) f(x_2).$$

For which $s \in \mathbb{R}$ can you say for sure that $g \notin H^s(\mathbb{T}^2)$?

Solution:

(a) We know that $f = \sum_{n=-\infty}^{\infty} \langle e_n, f\rangle e_n$, where $e_n(x) = (2\pi)^{-1/2}\exp(inx)$. For $n \neq 0$ we find

$$\begin{aligned}\langle e_n, f\rangle &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} e^{-inx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} \cos(nx)\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{n}\sin(nx)\right]_{-\pi/2}^{\pi/2} = \frac{1}{n\sqrt{2\pi}}\left(\sin(n\pi/2) - \sin(-n\pi/2)\right) = \frac{\sqrt{2}}{n\sqrt{\pi}}\sin(n\pi/2).\end{aligned}$$

For $n = 0$ we find

$$\langle e_0, f\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} dx = \sqrt{\pi/2}.$$

To summarize,

$$\begin{aligned} f(x) =& \sqrt{\pi/2}e_0(x) + \sum_{n=1,5,9,\dots}\frac{\sqrt{2}}{n\sqrt{\pi}}\big(e_n(x) + e_{-n}(x)\big) - \sum_{n=3,7,11,\dots}\frac{\sqrt{2}}{n\sqrt{\pi}}\big(e_n(x) + e_{-n}(x)\big) \\ =& \frac{1}{2} + \sum_{n=1,5,9,\dots}\frac{2}{n\pi}\cos(nx) - \sum_{n=3,7,11,\dots}\frac{2}{n\pi}\cos(nx).\end{aligned}$$

(Fully simplifying the formula was not required for full points.)

(b) For $s \geq 0$, we find that $||f||_{H^s}^2 = \sum_{n\in\mathbb{Z}}(1+|n|^2)^s|\langle e_n, f\rangle|^2 \sim \sum_{n\in\mathbb{N}} n^{2s}\frac{1}{n^2}$.
The sum is finite iff $2s - 2 < -1$, which is to say, if $s < 1/2$.

(c) Observe that $f \notin C^0(\mathbb{T}^2)$. Then the Sobolev embedding theorem tells us that $f \notin H^s(\mathbb{T}^2)$ if $s > 1$. (Since if $f \in H^s$ for $s > d/2 = 1$, then f would be continuous.)

For a more precise solution, you could use that

$$||f||_{H^s(\mathbb{T}^2)} = \sum_{n\in\mathbb{Z}^2}(1+|n|^2)^2|\langle e_n, f\rangle|^2,$$

and then use that for $n = (n_1, n_2)$ we have

$$\langle e_n, f\rangle = \frac{1}{2\pi}\int_{\mathbb{T}^2}\exp(i(n_1x_1 + n_2x_2))\,f(x)\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\exp(in_1x_1)\,dx_1\int_{-\pi/2}^{\pi/2}\exp(in_2x_2)\,dx_2.$$

Now use your results from part (a) to show $|\langle e_n, f\rangle| \sim 1/(1+|n_1|)(1+|n_2|)$ to get a precise result. (This precise solution was not required for full score.)

Problem 3: Set $I = [-1, 1]$ and let Ω denote the set of continuous functions on I, viewed as a subset of $H = L^2(I)$. Define an operator $A:\ \Omega \to L^2(I)$ via

$$[Au](x) = \frac{1}{2}u(x) + \frac{1}{2}u(-x).$$

(a) Prove that A can be uniquely extended to an operator in $\mathcal{B}(H)$.

(b) Is A a projection? If yes, is it an orthogonal projection?

Solution:

(a) We find that $\sup_{||u||=1} ||Au|| = \sup_{||u||=1} ||(1/2)u(x) + (1/2)u(-x)|| \leq \sup_{||u||=1} ((1/2)||u|| + (1/2)||u||) = ||u||$. This shows that A is continuous, and since Ω is dense, we know that there exists a unique extension.

(b) First we verify that A is a projection on Ω. Define a reflection operator R via $[Ru](x) = u(-x)$. Observe that $R^2 = I$. Then

$$A^2 = \big((1/2)I + (1/2)R\big)^2 = (1/4)I^2 + (1/2)R + (1/4)R^2 = (1/2)I + (1/2)R = A.$$

Since Ω is dense and A (and A^2) are continuous, the relationship $A^2 = A$ holds for the extended operator as well.

Next recall that A is orthogonal iff $||A|| = 0$ or $||A|| = 1$. We showed in (a) that $||A|| \leq 1$. To verify that $||A|| \geq 1$, simply observe that if $u = 1$ (or any even function), then $||Au|| = ||u||$. So yes, A is orthogonal.

Problem 4: Let $(e_n)_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H, and let $\mathcal{P}$ denote the set of all ***finite*** linear combinations of elements of e_n's. (Recall that we write this $\mathcal{P} = \text{Span}\big(e_n\big)_{n=1}^{\infty}$.) Prove that:

$$\mathcal{P} \text{ is dense} \qquad \Leftrightarrow \qquad \big(e_n\big)_{n=1}^{\infty} \text{ is an ON-basis.}$$

Solution: Suppose first that $(e_n)_{n=1}^{\infty}$ is a basis. Given any $f \in H$, define its partial expansion in (e_n) as usual:

$$f_N = \sum_{n=1}^{N} \langle e_n,\, f\rangle\, e_n \tag{1}$$

Since (e_n) is a basis, we know that $f_N \to f$ in norm. Since $f_N \in \mathcal{P}$, this proves that any function can be approximated arbitrarily well be functions in $\mathcal{P}$.

Suppose next that $\mathcal{P}$ is dense. Fix an $f \in H$, and define its partial expansion f_N as in (1). We need to prove that $f_N \to f$. Fix any $\varepsilon > 0$. Since $\mathcal{P}$ is dense, there is a $g \in \mathcal{P}$ such that $||f - g|| < \varepsilon$. Let N be a number such that $g \in \text{Span}(e_1,\, e_2,\, \dots,\, e_N) =: \mathcal{P}_N$. Now suppose that that $M \geq N$. Then since $g \in \mathcal{P}_M$, and f_M is the best possible approximant within $\mathcal{P}_M$, we find

$$||f - f_M|| \leq ||f - g|| < \varepsilon.$$

This shows that $f_N \to f$.