Homework set 14 — APPM5450, Spring 2014 — Solutions

Problem 12.8: We want to prove that

$$||f - f_n||_p^p = \int |f - f_n|^p \to \infty.$$

We know that $|f - f_n|^p \to 0$ pointwise, so if we can only justify moving the limit inside the integral, we'll be done.

First note that

$$|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq |g(x)|.$$

Then we can dominate the integrand as follows:

$$|f - f_n|^p \leq (|f| + |f_n|)^p \leq (|g| + |g|)^p \leq 2^p |g|^p.$$

Since $\int |g|^p < \infty$, we find that the Lebesque dominated convergence theorem applies, and so

$$\lim_{n\to\infty} ||f - f_n||_p^p = \lim_{n\to\infty} \int |f - f_n|^p = \{\text{LDCT}\} = \int \big(\lim_{n\to\infty} |f - f_n|^p\big) = \int 0 = 0.$$

Problem 12.16: Fix $f \in L^p$ and $\varepsilon > 0$. We want to prove that there exists a $\delta > 0$ such that for $|h| < \delta$, we have $||f - \tau_h f||_p < \varepsilon$.

First pick $\varphi \in C_c$ such that $||f - \varphi||_p < \varepsilon/3$. Then

$$\begin{aligned} ||f - \tau_h f||_p &\leq ||f - \varphi||_p + ||\varphi - \tau_h \varphi||_p + ||\tau_h \varphi - \tau_h f||_p \\ &= ||f - \varphi||_p + ||\varphi - \tau_h \varphi||_p + ||\varphi - f||_p < \varepsilon/3 + ||\varphi - \tau_h \varphi||_p + \varepsilon/3. \end{aligned}$$

Set $R = \sup\{|x| : \varphi(x) \neq 0\}$. Since φ is uniformly continuous, there exists a δ such that if $|x - y| < \delta$, then $|\varphi(x) - \varphi(y)| < \varepsilon/(3\mu(B_{R+1}(0))^{1/p})$. Then, if $h < \min(\delta, 1)$,

$$||\varphi - \tau_h \varphi||_p^p = \int_{B_{R+1}(0)} |\varphi(x) - \varphi(x - h)|^p \, dx < \int_{B_{R+1}(0)} \frac{\varepsilon^p}{3^p \mu(B_{R+1}(0))} \, dx < \frac{\varepsilon^p}{3^p}.$$

Problem 12.17: For $n = 1, 2, 3, \ldots$, set $I_n = (2^{-n}, 2^{-n+1})$, and $f_n = 2^{n/p} \chi_{I_n}$. Then $||f_n||_p = 1$ for all n. Suppose $m \neq n$, then

$$||f_n - f_m||_\infty = 1,$$

and for $p \in [1, \infty)$ we have

$$||f_n - f_m||_p = \left(\int_0^1 (2^n \chi_{I_n} + 2^m \chi_{I_m})\right)^{1/p} = 2^{1/p}.$$

No subsequence of $(f_n)_{n=1}^\infty$ can be Cauchy, and therefore no subsequence can converge.

Problem 12.18: For $n = 1, 2, 3, \ldots$, set $I_n = (2^{-n}, 2^{-n+1})$, and $f_n = 2^n \chi_{I_n}$. Let $(f_{n_j})_{j=1}^\infty$ be a subsequence of $(f_n)_{n=1}^\infty$. Define $g \in L^\infty$ by

$$g = \sum_{j=1}^\infty (-1)^j \chi_{I_{n_j}},$$

and define $\varphi \in (L^1)^*$ via $\varphi(f) = \int fg$. Then $\varphi(f_{n_j}) = (-1)^j$ (verify!) and so (f_{n_j}) cannot converge weakly. Since L^1 is not reflexive, this does not contradict that Banach-Alaoglu theorem.

Problem 12.13: Set $I = [0,\,1]$ and let Ω be a dense set in $L^\infty(I)$. For $r \in I$, set $f_r = \chi_{[0,\,r]}$, and pick $x_r \in \Omega \cap B_{1/3}(f_r)$. Since $||f_r - f_s|| = 1$ if $s \neq r$, we find that $||x_r - x_s|| \geq ||f_r - f_s|| - ||f_r - x_r|| - ||f_s - x_s|| \geq 1/3$, so all the x_r's are distinct. Therefore, Ω must be uncountable, and L^∞ cannot be seperable.

To prove that $C(I)$ cannot be dense in $L^\infty(I)$, simply note that if $f = \chi_{[0,1/2]}$, and $\varphi \in C(I)$, then

$$||f - \varphi||_\infty \geq \max(|\varphi(1/2)|,\, |1 - \varphi(1/2)|) \geq 1/2$$

(verify this!).

An alternative argument for why $C(I)$ cannot be dense in $L^\infty(I)$: If $\varphi_n \in C(I)$, and $\varphi_n \to f$ in the supnorm, then (φ_n) is a Cauchy sequence with respect to the uniform norm (when applied to continuous functions, the uniform norm and the L^∞ norms are identical). Therefore, there exists a continuous function φ such that $\varphi_n \to \varphi$ uniformly. Then $f(x) = \varphi(x)$ almost everywhere. But not every equivalence class function in L^∞ has a continuous function in it (for instance $f = \chi_{[0,1/2]}$).

Problem 12.14: Let p and q be such that $1 \leq p < q \leq \infty$.

First we construct a function $f \in L^p\backslash L^q$. Let α be a non-negative number and set $f(x) = x^{-\alpha}\chi_{[0,1]}$. Then

$$||f||_p^p = \int_0^1 x^{-\alpha\,p}\,dx,$$

which is finite if $\alpha p < 1$. Moreover

$$||f||_q^q = \int_0^1 x^{-\alpha\,q}\,dx$$

which is infinite if $\alpha q > 1$. Consequently, $f \in L^p\backslash L^q$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

To construct a function $f \in L^q\backslash L^p$, set $f = x^{-\alpha}\chi_{[1,\infty)}$. Then

$$||f||_p^p = \int_1^\infty x^{-\alpha p}\,dx$$

which is infinite if $\alpha p < 1$. Moreover

$$||f||_q^q = \int_1^\infty x^{-\alpha q}\,dx$$

which is finite if $\alpha q > 1$. Thus, $f \in L^1\backslash L^p$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

(The arguments above need slight modifications if $q = \infty$, but the idea is the same.)

Consider the function

$$f(x) = \frac{1}{\big(|x|\,(1 + \log^2|x|)\big)^{1/2}}.$$

That $f \in L^2$ is clear, since

$$||f||_2^2 = \int_{-\infty}^{\infty} \frac{1}{|x|(1+\log^2|x|)}\,dx = 2\int_0^{\infty} \frac{1}{x(1+\log^2 x)}\,dx = \{x = e^t\}$$

$$2\int_{-\infty}^{\infty} \frac{1}{e^t(1+t^2)} e^t\,dt = 2\pi.$$

Moreover, if $p > 2$, then note that there exists a $\delta > 0$ such that

$$x^{(p-2)/2}(1+\log^2 x)^{p/2} \leq 1$$

when $x \in (0,\,\delta)$. Then

$$||f||_p^p \geq \int_0^{\delta} \frac{1}{x^{p/2}(1+\log^2 x)^{p/2}}\,dx = \int_0^{\delta} \frac{1}{x}\underbrace{\frac{1}{x^{(p-2)/2}\,(1+\log^2 x)^{p/2}}}_{\geq 1}\,dx = \infty.$$

Analogously, if $p < 2$, then there exists an M such that

$$x^{(p-2)/2}(1+\log^2 x)^{p/2} \leq 1$$

when $x \geq M$. Then

$$||f||_p^p \geq \int_M^{\infty} \frac{1}{x^{p/2}(1+\log^2 x)^{p/2}}\,dx = \int_M^{\infty} \frac{1}{x}\underbrace{\frac{1}{x^{(p-2)/2}\,(1+\log^2 x)^{p/2}}}_{\geq 1}\,dx = \infty.$$

Problem 12.15: Let $\alpha \in (0,1)$, and let $m, n \in (1,\infty)$ be such that $1/m + 1/n = 1$ (we will determine suitable values for α, m, n later). Then from Hölder's inequality we obtain

$$(1) \qquad ||f||_r^r = \int |f|^r = \int |f|^{\alpha r}|f|^{(1-\alpha)r} \leq \left(\int |f|^{\alpha m r}\right)^{1/m}\left(\int |r|^{(1-\alpha)nr}\right)^{1/n}.$$

In order to obtain the desired right hand side, we must pick α, m, n so that

$$\begin{aligned} \alpha m r =& p, \\ (1-\alpha)nr =& q, \\ (1/m) + (1/n) =& 1. \end{aligned}$$

To obtain an equation for α, we eliminate m and n:

$$\frac{(1-\alpha)r}{q} = \frac{1}{n} = 1 - \frac{1}{m} = 1 - \frac{\alpha r}{p}.$$

Solving for α we obtain

$$\alpha = \frac{pq - pr}{rq - rp} = \frac{1/r - 1/q}{1/p - 1/q}.$$

Equation (1) now takes the form

$$||f||_r \leq \left(\left(||f||_p^p\right)^{1/m}\,\left(||f||_q^q\right)^{1/n}\right)^{1/r} = ||f||_p^{p/mr}\,||f||_q^{q/nr}.$$

Finally note that

$$\begin{aligned} \frac{p}{mr} =& \alpha = \frac{1/r - 1/q}{1/p - 1/q}, \\ \frac{q}{nr} =& 1 - \alpha = 1 - \frac{1/r - 1/q}{1/p - 1/q} = \frac{1/p - 1/r}{1/p - 1/q}. \end{aligned}$$

Problem 1: Let λ be a real number such that $\lambda \in (0,1)$, and let a and b be two non-negative real numbers. Prove that

$$a^{\lambda}\, b^{1-\lambda} \le \lambda\, a + (1-\lambda)\, b, \tag{2}$$

with equality iff $a = b$.

Solution: For $b = 0$ equation (2) reduces to $0 \le \lambda\, a$ which is clearly true.

When $b \neq 0$ we divide (2) by b and set $t = a/b$ to obtain

$$t^{\lambda} \le \lambda\, t + 1 - \lambda.$$

Set

$$f(t) = \lambda\, t + 1 - \lambda - t^{\lambda}.$$

We need to prove that $f(t) \ge 0$ when $t \ge 0$. First note that $f(0) = 1 - \lambda > 0$ and that $\lim_{t\to\infty} f(t) = \infty$. Since f is differentiable, we therefore need only investigate the points where $f'(t) = 0$. We find

$$f'(t) = \lambda - \lambda\, t^{\lambda-1}$$

so $f'(t) = 0$ happens only when $t = 1$. Now $f(1) = 0$ so it follows that $f(t) \ge 0$ for all $t \ge 0$, and that $f(t) = 0$ iff $t = 1$ (which is to say $a = b$).

Problem 2: [Hölder's inequality] Suppose that p is a real number such that $1 < p < \infty$, and let q be such that $p^{-1} + q^{-1} = 1$. Let (X, μ) be a measure space, and suppose that $f \in L^P(X,\mu)$ and $g \in L^q(X,\mu)$. Prove that $fg \in L^1(X,\mu)$, and that

$$||fg||_1 \le ||f||_p\, ||g||_q. \tag{3}$$

Prove that equality holds iff $\alpha|f|^p = \beta|g|^q$ for some α, β such that $\alpha\beta \neq 1$.

Solution: Suppose $||f||_p = 0$, then $f = 0$ a.e. and so (3) holds since both sides are identically zero. Analogously, (3) holds when $||g||_q = 0$.

Now suppose $||f||_p \neq 0$ and $||g||_q \neq 0$. Set

$$a = \left|\frac{f(x)}{||f||_p}\right|^p, \qquad b = \left|\frac{g(x)}{||g||_q}\right|^q, \qquad \lambda = \frac{1}{p}.$$

Then invoke (2), observing that $q(1-\lambda) = q(1 - 1/p) = q\,(1/q) = 1$, to obtain

$$\frac{|f(x)|}{||f||_p}\,\frac{|g(x)|}{||g||_q} \le \frac{1}{p}\,\frac{|f(x)|^p}{||f||_p^p} + \left(1 - \frac{1}{p}\right)\,\frac{|g(x)|^q}{||g||_q^q}.$$

Integrate over X to obtain

$$\frac{1}{||f||_p\,||g||_q}\int_X |f(x)|\,|g(x)|\,d\mu(x) \le \underbrace{\frac{1}{p}\,\frac{||f||_p^p}{||f||_p^p} + \left(1 - \frac{1}{p}\right)\,\frac{||g||_q^q}{||g||_q^q}}_{=1}.$$

Multiply by $||f||_p\,||g||_q$ to obtain (3).

Problem 3: [Minkowski's inequality] Let (X, μ) be a measure space, and let p be a real number such that $1 \le p \le \infty$. Prove that for $f, g \in L^p(X, \mu)$,

$$||f+g||_p \le ||f||_p + ||g||_p.$$

Solution:

Suppose $p = 1$:

$$||f+g||_1 = \int |f(x)+g(x)| \le \int \big(|f(x)| + |g(x)|\big) = \int |f(x)| + \int |g(x)| = ||f||_1 + ||g||_1.$$

Suppose $p = \infty$:

$$\begin{aligned} ||f+g||_\infty = \operatorname{ess\,sup} |f(x)+g(x)| &\le \operatorname{ess\,sup}\big(|f(x)| + |g(x)|\big) \\ &\le \operatorname{ess\,sup} |f(x)| + \operatorname{ess\,sup} |g(x)| = ||f||_\infty + ||g||_\infty. \end{aligned}$$

Suppose $p \in (1, \infty)$: The triangle inequality yields

$$|f(x)+g(x)|^p = |f(x)+g(x)|\,|f(x)+g(x)|^{p-1} \le \big(|f(x)| + |g(x)|\big)\,|f(x)+g(x)|^{p-1}.$$

Integrate both sides:

$$||f+g||_p^p \le \int |f(x)|\,|f(x)+g(x)|^{p-1} + \int |g(x)|\,|f(x)+g(x)|^{p-1}.$$

Now apply Hölder:

$$||f+g||_p^p \le ||f||_p\,||\,|f+g|^{p-1}||_q + ||g||_p\,||\,|f+g|^{p-1}||_q = \big(||f||_p + ||g||_p\big)\left(\int |f(x)+g(x)|^{q(p-1)}\right)^{1/q}.$$

Now use that $q = 1/(1-1/p) = p/(p-1)$ to see that $q(p-1) = p$ to get

$$||f+g||_p^p \le \big(||f||_p + ||g||_p\big)\left(\int |f(x)+g(x)|^p\right)^{1/q} = \big(||f||_p + ||g||_p\big)\,||f+g||_p^{p/q}.$$

Observe that $p/q = p(1-1/p) = p-1$ to obtain

$$||f+g||_p^p \le \big(||f||_p + ||g||_p\big)\,||f+g||_p^{p-1}$$

which gives Minkowski upon division by $||f+g||_p^{p-1}$.