Homework 11

11.12) Prove that if $s > n/2$, then $H^s(R^n) \subset C_0(R^n)$, and there is a constant C such that $\|f\|_\infty \le C\|f\|_{H^s}$ for all $f \in H^s(R^n)$

From the Riemann-Lebesgue lemma we know $\hat{f} \in L^1 \Rightarrow f \in C_0$ and $\|f\|_\infty \le (2\pi)^{n/2}\|\hat{f}\|_{L^1}$ (1)

Assume $f \in H^s$ for some $s > n/2$. Then $\|f\|_{H^s}^2 = \int |\hat{f}(t)|^2 (1+|t|^2)^s \, dt < \infty$

Also $\|\hat{f}\|_{L^1} = \int |\hat{f}(t)| dt = \int |\hat{f}(t)| (1+|t|^2)^{s/2} \frac{1}{(1+|t|^2)^{s/2}} dt \overset{CS}{\le} \left(\underbrace{\int |\hat{f}(t)|^2 (1+|t|^2)^s \, dt}_{=\|f\|_{H^s}^2} \right)^{1/2} \left(\int \frac{1}{(1+|t|^2)^s} dt \right)^{1/2} = *$

where the inequality denoted by "CS" uses Cauchy-Shwarz.

Looking at just the last part $\int_{R^n} \frac{1}{(1+|t|^2)^s} dt = C_n \int_0^\infty \frac{1}{(1+r^2)^s} r^{n-1} dr$

Note that C_n is the area of the unit sphere in R^n.

This integral is finite if $n-1-2s < -1$, which will hold if $s > n/2$.

Set $M = \left(\int \frac{1}{(1+|t|^2)^s} dt \right)^{1/2}$

Continuing from above we now have $\|\hat{f}\|_{L^1} \le * \le \|f\|_{H^s} M < \infty$, so $\hat{f} \in L^1$ (2)

Now the Riemann-Lebesgue lemma implies $f \in C_0$.

Combining (1) and (2) we now obtain $\|f\|_\infty \le \frac{1}{(2\pi)^{n/2}} \|\hat{f}\|_{L^1} \le \frac{1}{(2\pi)^{n/2}} M \|f\|_{H^s}$

11.19) Give a counterexample to show that the Riemann-Lebesgue lemma does not hold for all functions in L^2. That is, find a function $f \in L^2(R)$ such that $\hat{f}$ is not continuous.

We can easily do this by "working backwards."

Consider the non-continuous $\hat{f}(t) = \mathrm{X}_{[-1,1]}(t) = \begin{cases} 1 & -1 \le t \le 1 \\ 0 & else \end{cases}$. Then

$$f(x) = F^{-1}\left[\hat{f}\right](x) = \beta \int_{-\infty}^{\infty} e^{ixt} \mathrm{X}_{[-1,1]}(t) dt = \beta \int_{-1}^{1} e^{ixt} dt = \beta \left[\frac{1}{ix} e^{ixt} \right]_{-1}^{1} = \beta \left(\frac{e^{ix} - e^{-ix}}{ix} \right) = 2\beta \frac{\sin(x)}{x} = \sqrt{\frac{2}{\pi}} \frac{\sin(x)}{x}$$

Then $f \in L^2(R)$ and $\hat{f}$ is not continuous.

11.20) Show that $\delta \in H^s(R^n)$ if and only if $s < -n/2$.

$$\delta \in H^s \Leftrightarrow \hat{\delta}(t)\left(1+|t|^2\right)^{s/2} \in L^2 \overset{\hat{\delta}=\beta^n}{\Leftrightarrow} \beta^n \left(1+|t|^2\right)^{s/2} \in L^2 \Leftrightarrow \beta^n \int \left(1+|t|^2\right)^s dt < \infty \overset{Polar}{\Leftrightarrow} \beta^n \alpha^n \int_0^\infty \left(1+r^2\right)^s r^{n-1} dr < \infty \Leftrightarrow$$

$$\Leftrightarrow 2s + n - 1 < -1 \Leftrightarrow s < -n/2$$

Note that the iff denoted by "Polar" switches to polar coordinates and the iff across the line break uses $\int_0^\infty r^k dr < \infty \Leftrightarrow k < -1$.

11.21) Show that the integral equation $u(x)+\int_{-\infty}^{\infty} e^{-(x-y)^2/2}u(y)dy = f(x)$ (1) has a unique solution $u \in L^2(R)$ for every $f \in L^2(R)$, and give an expression for u in terms of f.

Existence

Set $\phi(x)=e^{\frac{-x^2}{2}}$ so that (1) takes the form $u+\phi * u = f$.

Then $\hat{f}(t)=\hat{u}(t)+\beta^{-n}\hat{\phi}(t)\hat{u}(t)=\left(1+\beta^{-n}\hat{\phi}(t)\right)\hat{u}(t) \Rightarrow \hat{u}(t)\frac{1}{1+\beta^{-n}\hat{\phi}(t)}\hat{f}(t)$

We need to verify that $u \in L^2$ when $f \in L^2$. Since F is an isometry we obtain:

$$\|u\|_{L^2}^2=\|\hat{u}\|_{L^2}^2=\int\frac{1}{\left(1+\beta^{-n}\hat{\phi}(t)\right)^2}\left|\hat{f}(t)\right|^2 dt \overset{*}{=} \int\frac{1}{\left(1+\beta^{-n}\phi(t)\right)^2}\left|\hat{f}(t)\right|^2 dt \le \int\left|\hat{f}(t)\right|^2 dt = \|f\|_{L^2}^2 < \infty$$

Note that the equality denoted by "*" uses $\beta^{-n}\hat{\phi}(t)=\beta^{-n}\phi(t)$ for this specific $\phi(t)$.

So $u \in L^2$ and $u(x)=\beta^n\int_{R^n} e^{ix\cdot t}\frac{\hat{f}(t)}{1+\beta^{-n}\hat{\phi}(t)}dt$.

Uniqueness

Assume u, v both solve the original problem.

Then $u+\phi * u = f$ and $v+\phi * v = f$. Subtraction yields $(u-v)+\phi*(u-v)=0$. Taking the Fourier transform of this yields $\left(1+\beta^n\hat{\phi}(t)\right)\left(\hat{u}(t)-\hat{v}(t)\right)=0$, which implies that $\hat{u}(t)=\hat{v}(t)$ almost everywhere.