APPM5450 — Applied Analysis: Final exam — Solutions
7:30pm – 10:00pm, May 7, 2014. Closed books.

Be smart in how you use your time. Some problems can potentially be finished very quickly — do these first. For instance, problems 1b,c,d and 2 should be fast. Problems 3 and 4 can in principle be solved quickly. Please motivate your answers unless the problem explicitly states otherwise.

Problem 1: (20p) The following problems are worth 4 points each. No motivation required.

(a) Which of the following operators are compact:

(i) $H = L^3(I)$ with $I = [0,1]$, and $[Au](x) = \int_0^1 \cos(x-y)\,u(y)\,dy$.

(ii) $H = L^2(\mathbb{R})$ and $[Au](x) = (1/2)\,u(x-1)$.

(iii) $H = L^2(\mathbb{Z}) = \ell^2(\mathbb{Z})$ and $[Au](n) = e^{-n^2}\,u(n)$.

(iv) $H = L^2(\mathbb{R})$ and $[Au](x) = e^{-x^2}\,u(x)$.

(b) State the Lebesgue dominated convergence theorem.

(c) State the Fatou lemma.

(d) Set $f_n(x) = n^{1/3}\,\chi_{(0,1/n)}$. Evaluate $||f_n||_p$ for $p \in [1,\infty)$ and specify what this information tells you about whether $(f_n)_{n=1}^{\infty}$ converges (weakly or strongly) in $L^p(\mathbb{R})$.

(e) Which of the following statements are necessarily correct for linear bounded operators on a Hilbert space H:

(i) If A is self-adjoint, then $B = \exp(iA)$ is unitary.

(ii) If A and B are self-adjoint, then $C = AB$ is also self-adjoint.

(iii) If A is self-adjoint, then A^2 is non-negative.

(iv) If A is skew-adjoint, then $B = (I-A)\,(I+A)^{-1}$ is unitary.

Solution:

(a) (i) and (iii) are compact.[1]

(d) $||f_n||_p = n^{1/3-1/p}$.
For $p > 3$, $||f_n||_p \to \infty$, so $(f_n)_{n=1}^{\infty}$ cannot converge either weakly or strongly.
For $p = 3$, we have $||f_n||_3 = 1$. This is not enough information to adjudicate convergence.
For $p < 3$, $||f_n||_p \to 0$, so $f_n \to 0$ both weakly and strongly.

(e) (i), (iii), (iv) are correct.[2]

Grading guide: -2p for each incorrect true/false answer.

[1](i) A is compact since $\mathrm{ran}(A)$ has dimension 2. (ii) A is not compact. Set for instance $f_n = \chi_{(n,n+1)}$. Then $f_n \rightharpoonup 0$, but (Af_n) does not converge in norm. (iii) A is compact. For any ε, set $M = \sqrt{\log(1/\varepsilon)}$ and $A_M = \chi_{[-M,M]}\,A$. Then $||A - A_M|| \leq \varepsilon$ and A_M is finite dimensional. (iv) $\sigma_{\rm c}(A) = [0,1]$. Recall that for a self-adjoint compact operator, the only possible point in the continuum spectrum is 0.

[2]For (ii), note that in general, operators do not commute. E.g., check $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 2: (20p) Recall that the Riemann-Lebesgue lemma states that if a function f is in $L^1(\mathbb{R}^d)$, then its Fourier transform $\hat{f}$ belongs to $C_0(\mathbb{R}^d)$. Please demonstrate how you can use this result to prove that if $f \in H^s(\mathbb{R}^d)$ for s "sufficiently high", then $f \in C_0(\mathbb{R}^d)$. Make sure to specify clearly what "sufficiently high" means.

Solution: Suppose that $f \in H^s(\mathbb{R}^d)$.

Observe that by the R-L Lemma, it is sufficient to prove that $\hat{f} \in L^1$ to establish that $f \in C_0$ (since $\mathcal{F}^{-1} = R\mathcal{F}$, the R-L lemma applies to $\mathcal{F}^{-1}$ too). To prove that $\hat{f} \in L^1$, we find

$$\int_{\mathbb{R}^d} |\hat{f}(t)| = \int_{\mathbb{R}^d} (1+|t|^2)^{-s/2}\,(1+|t|^2)^{s/2}\,|\hat{f}(t)| \leq \{\text{Cauchy-Schwartz}\} \leq$$
$$\leq \left(\int_{\mathbb{R}^d} (1+|t|^2)^{-s}\right)^{1/2} \left(\int_{\mathbb{R}^d} (1+|t|^2)^{s}\,|\hat{f}(t)|^2\right)^{1/2} = \left(\int_{\mathbb{R}^d} (1+|t|^2)^{-s}\right)^{1/2} ||f||_{H^s}$$

By switching to polar coordinates, we can verify when the right hand side is finite

$$\int_{\mathbb{R}^d} (1+|t|^2)^{-s} dt = S_d \int_0^\infty (1+r^2)^{-s}\, r^{d-1}\, dr,$$

where S_d is the area of the unit sphere in $\mathbb{R}^d$. It is clear that this integral is finite iff $-2s+d-1 < -1$, which is to say: $\boxed{\text{If } s > d/2, \text{ then } \hat{f} \in L^1, \text{ and therefore } f \in C_0.}$

Problem 3: (20p) Specify $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, $\sigma_{\rm r}(A)$ for the following operators:

(a) $H = L^2(\mathbb{R})$ and $[Au](x) = u(x) + u(-x)$.

(b) $H = L^2(\mathbb{Z})$ and $[Au](n) = e^{-n^2}\,u(n)$.

(c) $H = L^2(\mathbb{R})$ and $[Au](x) = [\mathcal{F}u](x)$ (Fourier transform).

(d) $H = L^2(\mathbb{R})$ and $[Au](x) = u(x-1)$.

Solution: Only the non-empty parts of the spectra are given.

(a) $\sigma_{\rm p}(A) = \{0, 2\}$.

(Observe that $A = 2P$, where P is the projection onto the even functions.)

(b) $\sigma_{\rm p}(A) = \{e^{-n^2}\}_{n=1}^{\infty}$, $\sigma_{\rm c}(A) = \{0\}$.

(Observe that for each n, e^{-n^2} is an eigenvalue associated with the canonical basis vector e_n. Then since the spectrum has to be closed, the cluster point 0 of $\sigma_{\rm p}(A)$ must also be included. This point must be in the continuum spectrum since A is self-adjoint, and therefore $\sigma_{\rm r}(A) = \emptyset$.)

(c) $\sigma_{\rm p}(A) = \{1,\, -1,\, i,\, -i\}$.

($\mathcal{F}$ is unitary, which immediately tells you that the spectrum is contained in the unit circle. Then from the relation $\mathcal{F}^4 = I$, you find that any $\lambda \in \sigma(A)$ must satisfy $\lambda^4 = 1$. Verifying that each is the possible solutions actually is an eigenvalue is a little trickier.)

(d) $\sigma_{\rm c}(A) = \{\lambda \in \mathbb{C} :\ |\lambda| = 1\}$.

(A is unitary, which immediately tells you that the spectrum is contained in the unit circle. To verify the precise statement, move to Fourier space, and observe that, with $\hat{A} = \mathcal{F}A\mathcal{F}^*$, we have $[\hat{A}\,\hat{u}](t) = e^{-it}\,\hat{u}(t)$, in other words, $\hat{A}$ is a diagonal operator and so $\sigma(A) = \sigma(\hat{A}) = \overline{\{e^{-it} :\ t \in \mathbb{R}\}}$.)

Problem 4: (20p) Let $\mathcal{S}(\mathbb{R})$ denote the set of Schwartz functions as usual, and define for $n = 1, 2, 3, \ldots$ a linear function T_n on $\mathcal{S}(\mathbb{R})$ via

$$\langle T_n, \varphi\rangle = \int_{-\infty}^{-1/n} \frac{1}{x}\,\varphi(x)\,dx + \int_{1/n}^{\infty} \frac{1}{x}\,\varphi(x)\,dx.$$

(a) (5p) Prove that each T_n is a continuous map $T_n : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$. What is the order of T_n? (Recall that the *order* of a distribution U is the lowest number m for which a bound of the form $|U(\varphi)| \leq C\sum_{|\alpha|\leq m}\sum_{\ell\leq k}||\varphi||_{\alpha,\ell}$ holds. It measures how many *derivatives* in φ you need to bound U.)

(b) (10p) Prove that there exists a continuous functional T such that $T_n \to T$ in $\mathcal{S}^*(\mathbb{R})$.

(c) (5p) Specify the Fourier transform $\hat{T}$ of T. No motivation required.
Hint: You may want to try to determine the product $x\,T$.

Solution:

(a) Each T_n is a bounded linear functional of order 0:

$$\begin{aligned}|T_n(\varphi)| &\leq \int_{1/n}^{1} \frac{1}{x}(|\varphi(x)| + |\varphi(-x)|)\,dx + \int_{1}^{\infty} \frac{1}{x}(|\varphi(x)| + |\varphi(-x)|)\,dx \\ &\leq \int_{1/n}^{1} \frac{1}{x}2||\varphi||_{0,0}\,dx + \int_{1}^{\infty} \frac{1}{x}\frac{1}{(1+x^2)^{1/2}}2||\varphi||_{0,1}\,dx \leq 2\log(n)\,||\varphi||_{0,0} + 2||\varphi||_{0,1}.\end{aligned}$$

(b) For each n, we have

$$T_n(\varphi) = \int_{1/n}^{\infty} \frac{\varphi(x) - \varphi(x)}{x}\,dx.$$

Observe that for $0 < x \leq 1$, we have, for any $\varphi \in \mathcal{S}$,

$$\left|\frac{\varphi(x) - \varphi(-x)}{x}\right| = \left|\frac{1}{x}\int_{-x}^{x}\varphi'(t)\,dt\right| \leq 2\,||\varphi||_{1,0}. \tag{1}$$

For $1 \leq x$, we find

$$\left|\frac{\varphi(x) - \varphi(-x)}{x}\right| \leq \frac{|\varphi(x)| + |\varphi(-x)|}{x} \leq \frac{2||\varphi||_{0,1}}{(1+x^2)^{1/2}x} \leq \frac{2||\varphi||_{0,1}}{x^2}. \tag{2}$$

Define a functional T via

$$T(\varphi) = \int_{0}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x}\,dx.$$

It follows from (1) and (2) that T is well-defined, and that $|T(\varphi)| \leq 2||\varphi||_{0,1} + 2||\varphi||_{1,0}$, so $T \in \mathcal{S}^*$. Finally, observe that for any fixed φ,

$$|T(\varphi) - T_n(\varphi)| = \left|\int_{0}^{1/n} \frac{\varphi(x) - \varphi(-x)}{x}\,dx\right| \leq \int_{0}^{1/n} 2||\varphi||_{1,0}\,dx = \frac{1}{n}2||\varphi||_{1,0} \to 0, \qquad \text{as } n \to \infty.$$

(c) $\hat{T}(x) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sign}(x)$.

For a detailed solution, see the posted solution to problem 11.22 in the textbook.

To get a clue to what the answer *might* be, note $xT = 1$, and so $-ixT = -i$. Then $\partial_t\hat{T} = -i\sqrt{2\pi}\delta$. Integrating, we see that $\hat{T}$ should be a step function with a jump of size $-i\sqrt{2\pi}$ at the origin.

Grading guide: In (a), you lose 2 points if you bound the order by 1 instead of 0.

Problem 5: (20p) Consider for $p \in [1, \infty)$ the Banach space $L^p(\mathbb{R})$. Define a functional φ on the subspace $C_c(\mathbb{R})$ via

$$\varphi(f) = \int_1^\infty \frac{1}{\sqrt{x}}\, f(x)\, dx.$$

Recall that $C_c(\mathbb{R})$, the set of compactly supported continuous functions, is dense in $L^p(\mathbb{R})$.

For which $p \in [1, \infty)$, if any, can φ be extended to a continuous linear functional on all of $L^p(\mathbb{R})$?

For any p for which you claim that $\varphi \in (L^p)^*$, give an upper bound for $||\varphi||_{(L^p)^*}$.

Solution: Set $g(x) = x^{-1/2}\, \chi_{[1,\infty)}$. Then $||g||_q^q = \displaystyle\int_1^\infty x^{-q/2}$. We find that $g \in L^q$ iff $q > 2$.

In this problem, view q as a function of p, $q = p/(p-1)$ so that $1/p + 1/q = 1$.

Case 1 — $p \in [1, 2)$: In this case $\varphi \in (L^p(\mathbb{R}))^*$ since by the Hölder inequality $|\varphi(f)| \leq ||g||_q ||f||_p$, and $||g||_q$ is finite. Moreover, $||\varphi||_{(L^p)^*} \leq ||g||_{L^q} = (2/(q-2))^{1/q} = ((2p-2)/(2-p))^{1-1/p}$.

Case 2 — $p = 2$: Pick α such that $\alpha > 1/2$ and set $f_\alpha(x) = x^{-\alpha}\, \chi_{[1,\infty)}$. Then $f_\alpha \in L^2$, and $||f_\alpha||_2^2 = \int_1^\infty x^{-2\alpha}\, dx = (2\alpha - 1)^{-1}$. Moreover, $\varphi(f_\alpha) = \int_1^\infty x^{-1/2-\alpha}\, dx = 2(2\alpha-1)^{-1}$. Now

$$||\varphi|| = \sup_{f \neq 0} \frac{|\varphi(f)|}{||f||_2} \geq \sup_{\alpha > 1/2} \frac{|\varphi(f_\alpha)|}{||f_\alpha||_2} = \sup_{\alpha > 1/2} \frac{2(2\alpha-1)^{-1}}{(2\alpha-1)^{-1/2}} = \sup_{\alpha > 1/2} 2(2\alpha-1)^{-1/2} = \infty.$$

We see that φ cannot be in the dual of L^2.

Case 3 — $p > 2$: Pick α such that $1/p < \alpha < 1/2$ and set $f(x) = x^{-\alpha}\, \chi_{[1,\infty)}$. Then $f \in L^p$, but $\varphi(f) = \infty$, so φ cannot be a continuous linear functional on L^p.

Grading guide: Points are allocated as follows:
12p for a perfect answer to the case $p \in [1, 2)$.
3p for a perfect answer to the case $p = 2$.
5p for a perfect answer to the case $p \in (2, \infty)$.

Note that Hölder's inequality immediately provides an upper bound for $||\varphi||_{(L^p)^*}$. finding a maximizer, or a maximizing sequence, in the expression $||\varphi||_{(L^p)^*} = \sup_{||f||_p = 1} |\varphi(f)|$ is a harder. This was *not* required to receive full points.