APPM5450 — Applied Analysis: Final exam

7:30pm – 10:00pm, May 7, 2014. Closed books.

Be smart in how you use your time. Some problems can potentially be finished very quickly — do these first. For instance, problems 1b,c,d, 2, and 3 should be fast. Please motivate your answers unless the problem explicitly states otherwise.

Problem 1: (20p) The following problems are worth 4 points each. No motivation required.

(a) Which of the following operators are compact:
  (i) $H = L^3(I)$ with $I = [0, 1]$, and $[Au](x) = \int_0^1 \cos(x - y)\, u(y)\, dy$.
  (ii) $H = L^2(\mathbb{R})$ and $[Au](x) = (1/2)\, u(x - 1)$.
  (iii) $H = L^2(\mathbb{Z}) = \ell^2(\mathbb{Z})$ and $[Au](n) = e^{-n^2}\, u(n)$.
  (iv) $H = L^2(\mathbb{R})$ and $[Au](x) = e^{-x^2}\, u(x)$.

(b) State the Lebesgue dominated convergence theorem.

(c) State the Fatou lemma.

(d) Set $f_n(x) = n^{1/3}\, \chi_{(0,1/n)}$. Evaluate $||f_n||_p$ for $p \in [1, \infty)$ and specify what this information tells you about whether $(f_n)_{n=1}^{\infty}$ converges (weakly or strongly) in $L^p(\mathbb{R})$.

(e) Which of the following statements are necessarily correct for linear bounded operators on a Hilbert space H:
  (i) If A is self-adjoint, then $B = \exp(iA)$ is unitary.
  (ii) If A and B are self-adjoint, then $C = AB$ is also self-adjoint.
  (iii) If A is self-adjoint, then A^2 is non-negative.
  (iv) If A is skew-adjoint, then $B = (I - A)\,(I + A)^{-1}$ is unitary.

Problem 2: (20p) Recall that the Riemann-Lebesgue lemma states that if a function f is in $L^1(\mathbb{R}^d)$, then its Fourier transform $\hat{f}$ belongs to $C_0(\mathbb{R}^d)$. Please demonstrate how you can use this result to prove that if $f \in H^s(\mathbb{R}^d)$ for s "sufficiently high", then $f \in C_0(\mathbb{R}^d)$. Make sure to specify clearly what "sufficiently high" means.

Problem 3: (20p) Specify $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, $\sigma_{\rm r}(A)$ for the following operators:

(a) $H = L^2(\mathbb{R})$ and $[Au](x) = u(x) + u(-x)$.

(b) $H = L^2(\mathbb{Z})$ and $[Au](n) = e^{-n^2}\, u(n)$.

(c) $H = L^2(\mathbb{R})$ and $[Au](x) = [\mathcal{F}u](x)$ (Fourier transform).

(d) $H = L^2(\mathbb{R})$ and $[Au](x) = u(x - 1)$.

No motivation required. If you cannot answer a problem fully, then please give what information you can about the spectrum.

Problem 4: (20p) Let $\mathcal{S}(\mathbb{R})$ denote the set of Schwartz functions as usual, and define for $n = 1, 2, 3, \ldots$ a linear function T_n on $\mathcal{S}(\mathbb{R})$ via

$$\langle T_n, \varphi \rangle = \int_{-\infty}^{-1/n} \frac{1}{x}\,\varphi(x)\,dx + \int_{1/n}^{\infty} \frac{1}{x}\,\varphi(x)\,dx.$$

(a) (5p) Prove that each T_n is a continuous map $T_n : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$. What is the order of T_n? (Recall that the *order* of a distribution U is the lowest number m for which a bound of the form $|U(\varphi)| \leq C \sum_{|\alpha| \leq m} \sum_{\ell \leq k} ||\varphi||_{\alpha,\ell}$ holds. It measures how many *derivatives* in φ you need to bound U.)

(b) (10p) Prove that there exists a continuous functional T such that $T_n \to T$ in $\mathcal{S}^*(\mathbb{R})$.

(c) (5p) Specify the Fourier transform $\hat{T}$ of T. No motivation required.
Hint: You may want to try to determine the product $x\,T$.

Problem 5: (20p) Consider for $p \in [1, \infty)$ the Banach space $L^p(\mathbb{R})$. Define a functional φ on the subspace $C_{\mathrm{c}}(\mathbb{R})$ via

$$\varphi(f) = \int_1^{\infty} \frac{1}{\sqrt{x}}\,f(x)\,dx.$$

Recall that $C_{\mathrm{c}}(\mathbb{R})$, the set of compactly supported continuous functions, is dense in $L^p(\mathbb{R})$.

For which $p \in [1, \infty)$, if any, can φ be extended to a continuous linear functional on all of $L^p(\mathbb{R})$?

For any p for which you claim that $\varphi \in (L^p)^*$, give an upper bound for $||\varphi||_{(L^p)^*}$.