APPM5450 — Applied Analysis: Section exam 1 — Solutions
8:30 – 9:50, February 25, 2013. Closed books.

Problem 1: Let H be a Hilbert space with an ON-basis $(e_j)_{j=1}^{\infty}$.

(a) State what it means for a sequence $(u_n)_{n=1}^{\infty}$ in H to *converge weakly* to a vector $u \in H$.

(b) Suppose that you are given a sequence of vectors $(u_n)_{n=1}^{\infty}$ for which you know:
(1) There exists a finite M such that $||u_n|| \leq M$ for every n.
(2) There is a vector $u \in H$ such that for every j, we have $\lim_{n\to\infty}(e_j, u_n) = (e_j, u)$.
Is is necessarily the case that (u_n) converges weakly to u? Either prove directly from the definition you gave in (a) that this is true, or give a counter-example.

Solution:

(a) For every $y \in H$, it is the case that $\lim_{n\to\infty}(y, u_n) = (y, u)$.

(b) Suppose (1) and (2) hold.

Pick $y \in H$. Fix $\varepsilon > 0$.

Pick $y' \in \text{Span}\{e_j\}_{j=1}^{\infty}$ such that $||y - y'|| < \varepsilon/(M + ||u||)$.

Now observe that

$$\begin{aligned}
\limsup_{n\to\infty} |(y, u_n - u)| &= \limsup_{n\to\infty} |(y', u_n - u) + (y - y', u_n - u)| \\
&\leq \limsup_{n\to\infty} \big(|(y', u_n - u)| + ||y - y'||\, ||u_n - u||\big) \\
&\leq \limsup_{n\to\infty} \big(|(y', u_n - u)| + ||y - y'||\, (M + ||u||)\big) \\
&= ||y - y'||\, (M + ||u||\big) < \varepsilon.
\end{aligned}$$

(The last equality follows from assumption (2) since y' is a finite linear combination of e_j's.)

Problem 2: Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a self-adjoint operator. Let b be a non-zero real number. Prove that the operator $B = A + i\,b\,I$ has closed range (where "i" is the imaginary unit). Is B necessarily one-to-one? Is B necessarily onto?

Solution: First we prove that B is necessarily coercive. For any $x \in H$, we have

$$||Bx||^2 = (Ax + ibx, Ax + ibx) = ||Ax||^2 + 2\text{Re}\big((Ax, ibx)\big) + ||ibx||^2 \geq 2\text{Re}\big((Ax, ibx)\big) + b^2||x||.$$

Since A is self-adjoint, the number $(Ax, ibx) = ib\,(Ax, x)$ is a purely imaginary. Therefore

$$||Bx||^2 \geq b^2||x||^2.$$

Since B is coercive, it necessarily has closed range due to the closed range theorem.

Since B is coercive, it must obviously be one-to-one.

To prove that B is onto, note that

$$\overline{\text{ran}(B)} = \ker(B^*)^\perp = \ker(A^* - ibI)^\perp = \ker(A - ibI)^\perp.$$

We proved above that $A - ibI$ is coercive, so

$$\overline{\text{ran}(B)} = \{0\}^\perp = H.$$

Finally invoke our finding that B has closed range to deduce $\text{ran}(B) = H$.

Problem 3: Set $I = [-\pi,\, \pi]$, and consider for $n = 1,\, 2,\, 3,\, \dots$ the functions

$$u_n(x) = \sum_{j=1}^{n} \frac{1}{j^{7/4}}\, \cos\big((2j-1)x\big).$$

(a) Does the sequence $(u_n)_{n=1}^\infty$ converge in $L^2(I)$? In $C(I)$? In $C^1(I)$? In $H^k(I)$ for any positive k? Please motivate your answers briefly.

(b) What can you tell about the sequence $(u'_n)_{n=1}^\infty$ of *derivatives* of u_n's? Does it converge in any of the spaces mentioned in part (a)?

Solution:

(a) First observe that (u_n) is a Cauchy sequence in $C(I)$ since, for $N \le m \le n$, we have

$$|u_n(x) - u_m(x)| = \left| \sum_{j=m+1}^{n} \frac{1}{j^{7/4}}\, \cos\big((2j-1)x\big) \right| \le \sum_{j=m+1}^{n} \frac{1}{j^{7/4}} \le \sum_{j=N+1}^{\infty} \frac{1}{j^{7/4}}.$$

Consequently, the limit function

$$u(x) = \sum_{j=1}^{\infty} \frac{1}{j^{7/4}}\, \cos\big((2j-1)x\big)$$

exists, and it is a continuous function. $u_n \to u$ in $C(I)$, and hence also in $L^2(I)$.

Next we check in which Sobolev spaces we have convergence. Let $\alpha_n = (e_n, u)$ be the n'th Fourier coefficient of u. By direct inspection, we find

$$\alpha_n = \begin{cases} 0 & n \text{ is even} \\ \sqrt{\frac{\pi}{2}}\, \left(\frac{2}{|n|+1}\right)^{7/4} & n \text{ is odd.} \end{cases}$$

For $k \ge 0$, we have

$$||u||_{H^k}^2 = \sum_{n=-\infty}^{\infty} (1 + |n|^2)^k\, |\alpha_n|^2 \sim \sum_{n=1}^{\infty} n^{2k}\, \frac{1}{n^{7/2}}.$$

The sum is finite if and only if $2k - 7/2 < -1$. In other words,

$$u \in H^k \qquad \Leftrightarrow \qquad k < 5/4.$$

So $u_n \to u$ in $H^k(I)$ for $k < 5/4$. By the Sobolev embedding theorem, we find that u does not converge in $C^1(I)$ (since $u \notin H^{3/2+\epsilon}$).

(b) Since

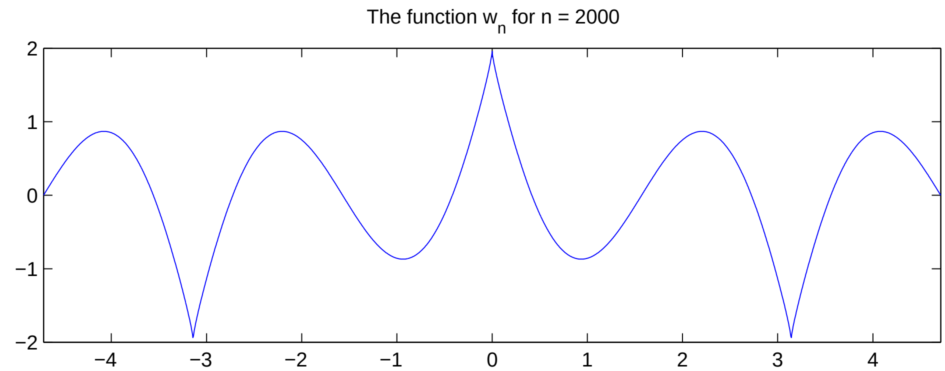
$$u \in H^k \qquad \Leftrightarrow \qquad u' \in H^{k-1},$$

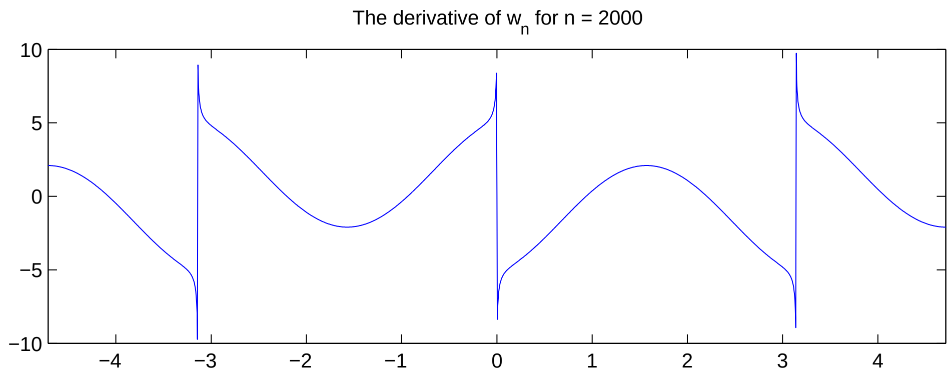
we find that

$$u' \in H^k \qquad \Leftrightarrow \qquad k < 1/4.$$

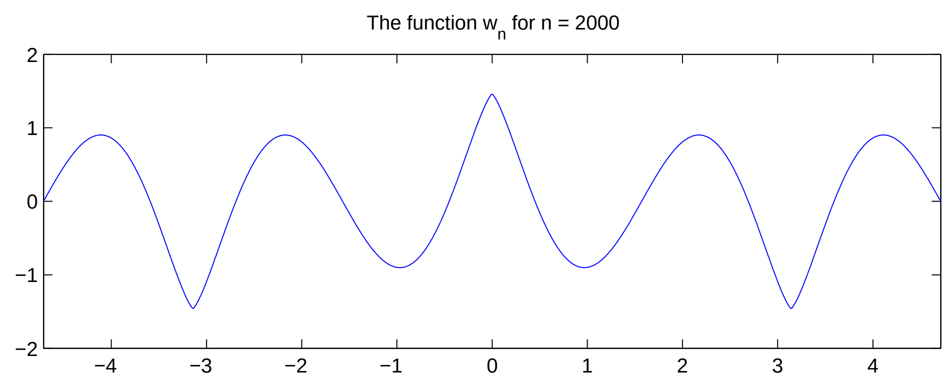
Therefore, $(u'_n)_{n=1}^\infty$ converges in $L^2(I)$ and in $H^k(I)$ when $k < 1/4$.
But $(u'_n)_{n=1}^\infty$ converges in neither $C(I)$ nor $C^1(I)$.

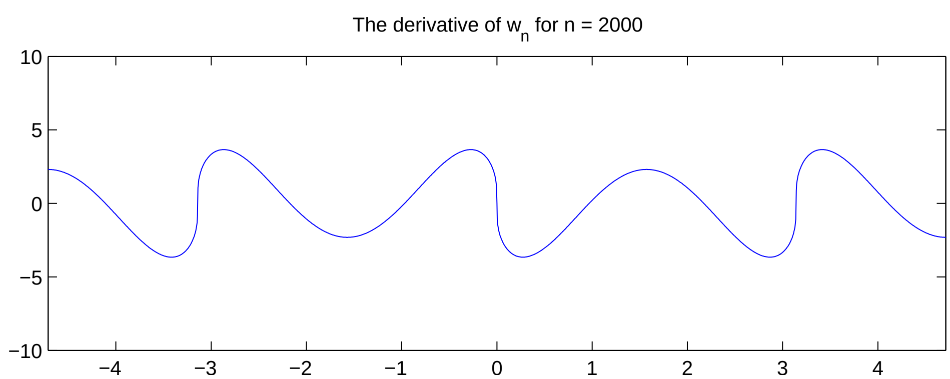
Plots of the function u_n for large n. Note that u_n looks to be continuous, but not C^1.





Next we plot $w_n(x) = \sum_{j=1}^{n} \frac{1}{j^{9/4}} \cos\big((2j-1)x\big)$. A little faster decay in the Fourier coefficients gives us a little more smoothness — just enough to push us into C^1 continuity.





Problem 4: Set $I = [-\pi, \pi]$, $H = L^2(I)$, and let $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ denote the standard Fourier basis for H. Define for $t \geq 0$, the operator $A(t)$ via

$$A(t)u = \sum_{n=-\infty}^{\infty} e^{-n^2 t}\,(e_n, u)\,e_n.$$

(a) Prove that for any $t \geq 0$, the operator $A(t)$ is continuous and determine $||A(t)||$.

(b) Prove that for any $t > 0$ and for any $u \in H$, it is the case that $A(t)\,u \in C^k(I)$ for any k.

(c) Fix $u \in H$ and define for $t > 0$ the function $v(t, x) = A(t)u$. State a second order partial differential equation that v satisfies (with boundary conditions). No motivation required.

(d) Define for $m = 1, 2, 3, \ldots$ the operator $B_m = A(1/m)$. Does $(B_m)_{m=1}^{\infty}$ converge in $\mathcal{B}(H)$? If so, to what? In what sense? No motivation required.

Solution:

(a) For any $u \in H$ and for any $t \geq 0$, we have

$$||Au||^2 = \sum_n e^{-2n^2 t}|(e_n, u)|^2 \leq \sum_n |(e_n, u)|^2 = ||u||^2,$$

so $||A|| \leq 1$. To prove that $||A|| = 1$, simply note that $Ae_0 = e_0$.

(b) For any $t > 0$ and for any $k \geq 0$, set

$$C = \sup_{n \in \mathbb{Z}} (1 + |n|^2)^k e^{-2n^2 t}.$$

Clearly C must be finite. Now for any $u \in H$, we have

$$||A(t)u||_{H^k}^2 = \sum_n (1 + |n|^2)^k e^{-2n^2 t}|(e_n, u)|^2 \leq C \sum_n |(e_n, u)|^2 = C\,||u||^2,$$

so $A(t)u \in H^k$ for any k. From the Sobolev embedding theorem we get $A(t)u \in C^k$ for any k.

(c) v satisfies the heat equation with periodic boundary conditions:

$$\Delta v = \frac{\partial^2 v}{\partial t^2}$$
$$v(-\pi) = v(\pi)$$
$$v'(-\pi) = v'(\pi)$$

The initial condition is

$$v(0, x) = u(x),$$

but this is enforced only in the sense that $\lim_{t \to 0} ||v(t, \cdot) - u||_{L^2(I)} = 0$.
(No discussion of the initial condition is required for full credit.)

(d) (B_m) converges strongly to the identity operator. It does not converge in norm.

First we prove that $B_m \to I$ strongly. Fix $u \in H$. Fix $\varepsilon > 0$. Pick N such that

$$\sum_{|n| > N} |(e_n, u)|^2 < \varepsilon.$$

Then

$$\begin{aligned}\limsup_{n\to\infty} ||B_m u - u||^2 &= \limsup_{m\to\infty}\left(\sum_{|n|\le N} |e^{-n^2/m} - 1|^2 |(e_n, u)|^2 + \sum_{|n|>N} |e^{-n^2/m} - 1|^2 |(e_n, u)|^2\right) \\ &\le \limsup_{m\to\infty}\left(\sum_{|n|\le N} |e^{-n^2/m} - 1|^2 |(e_n, u)|^2 + \sum_{|n|>N} |(e_n, u)|^2\right) \\ &\le \limsup_{m\to\infty}\left(\sum_{|n|\le N} |e^{-n^2/m} - 1|^2 |(e_n, u)|^2 + \varepsilon^2\right) = \varepsilon^2.\end{aligned}$$

Since ε is arbitrary, we find $||B_m u - u|| \to 0$.

To see that B_m does not converge in norm, observe that I is the only possibly limit point (since the sequence converges strongly to I), and that

$$||B_m - I|| \ge \sup_n ||B_m e_n - e_n|| = \sup_n |e^{-n^2/m} - 1| = 1.$$