Solutions for Homework 9 — APPM5450 — Spring 2013

11.5: Note that
$$\frac{1}{x+i\varepsilon} = \frac{x}{\varepsilon^2+x^2} - i\frac{\varepsilon}{\varepsilon^2+x^2}.$$
Fix a $\varphi \in \mathcal{S}$. You need to prove that

(1) $$\lim_{\varepsilon\to 0}\langle i\frac{\varepsilon}{\varepsilon^2+x^2},\, \varphi\rangle \to -i\pi\varphi(0).$$

and that

(2) $$\lim_{\varepsilon\to 0}\langle \frac{x}{\varepsilon^2+x^2},\, \varphi\rangle \to \langle \mathrm{PV}\left(\frac{1}{x}\right),\, \varphi\rangle,$$

Proving (1) is simple:
$$\langle i\frac{\varepsilon}{\varepsilon^2+x^2},\, \varphi\rangle = \int_{-\infty}^{\infty} i\frac{\varepsilon}{\varepsilon^2+x^2}\,\varphi(x)\,dx = \{\text{Set } x = \varepsilon y\} = \cdots$$
For (2) we need to work a bit more (unless I overlook a simpler solution)
$$\begin{aligned}
&\lim_{\varepsilon\to 0}\langle \frac{x}{\varepsilon^2+x^2},\, \varphi\rangle - \langle \mathrm{PV}\left(\frac{1}{x}\right),\, \varphi\rangle \\
&\qquad = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{x}{\varepsilon^2+x^2}\,\varphi(x)\,dx - \lim_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\frac{1}{x}\,\varphi(x)\,dx \\
&\qquad = \underbrace{\lim_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\left(\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right)\varphi(x)\,dx}_{=S_1} + \underbrace{\lim_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}}\frac{x}{\varepsilon^2+x^2}\,\varphi(x)\,dx}_{=S_2}.
\end{aligned}$$
First we bound $|S_1|$. Note that when $|x| \geq \sqrt{\varepsilon}$, we have
$$\left|\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right| = \frac{\varepsilon^2}{|x|(\varepsilon^2+x^2)} \leq \frac{\varepsilon^2}{|x|^3} \leq \frac{\varepsilon^2}{\varepsilon^{3/2}} = \sqrt{\varepsilon}.$$
Consequently,
$$\begin{aligned}
|S_1| &\leq \limsup_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\left|\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right|\,|\varphi(x)|\,dx \\
&\leq \limsup_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\sqrt{\varepsilon}\frac{1}{(1+|x|^2)}\underbrace{|(1+|x|^2)\varphi(x)|}_{\leq||\varphi||_{0,2}}\,dx = 0.
\end{aligned}$$
In bounding S_2 we use that
$$\int_{|x|\leq\sqrt{\varepsilon}}\frac{x}{\varepsilon^2+x^2}\,\varphi(0)\,dx = 0,$$
and that
$$|\varphi(x) - \varphi(0)| \leq |x|\,||\varphi'||_{\mathrm{u}} \leq |x|||\varphi||_{1,0},$$
to obtain
$$\begin{aligned}
|S_2| &= \left|\lim_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}}\frac{x}{\varepsilon^2+x^2}\,(\varphi(x)-\varphi(0))\,dx\right| \\
&\leq \limsup_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}}\underbrace{\frac{|x|}{\varepsilon^2+x^2}\,|x|}_{\leq 1}\,||\varphi||_{1,0}\,dx = 0.
\end{aligned}$$

Problem 11.6: We find that

$$\langle D(\log|x|)\,\varphi\rangle = -\langle \log|x|\,\varphi'\rangle = -\int_{\mathbb{R}} \log|x|\,\varphi'(x)\,dx$$
$$= -\lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{-\varepsilon} \log(-x)\varphi'(x)\,dx + \int_{\varepsilon}^{\infty} \log(x)\varphi'(x)\,dx\right\}.$$

Partial integrations yield

$$-\langle \log|x|\,\varphi'\rangle = -\lim_{\varepsilon\to 0}\Big\{[\log(-x)\varphi(x)]_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon} \frac{1}{-x}\varphi(x)\,dx +$$
$$[\log(x)\varphi(x)]_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{1}{x}\varphi(x)\,dx\Big\}$$
$$= \langle \mathrm{PV}(1/x),\,\varphi\rangle + \lim_{\varepsilon\to 0}\left\{\log(\varepsilon)\big(\varphi(\varepsilon)-\varphi(-\varepsilon)\big)\right\}.$$

Since

$$\big|\varphi(\varepsilon)-\varphi(-\varepsilon)\big| = \left|\int_{-\varepsilon}^{\varepsilon} \varphi'(x)\,dx\right| \leq 2\varepsilon||\varphi||_{1,0}$$

and $\lim_{\varepsilon\to 0}\{\varepsilon\log\varepsilon\} = 0$, we find that $\lim_{\varepsilon\to\ 0}\left\{\log(\varepsilon)\big(\varphi(\varepsilon)-\varphi(-\varepsilon)\big)\right\} = 0$.

Problem 11.7: First prove that $x\cdot\delta(x) = 0$ and that $x\cdot\mathrm{PV}(1/x) = 1$ (using the regular rules for the product between a polynomial and a Schwartz function). Suppose that $\cdot$ is distributive and can pair any two distributions. Then on the one hand we would have

$$\delta(x)\cdot x\cdot\mathrm{PV}(1/x) = \delta(x)\cdot(x\cdot\mathrm{PV}(1/x)) = \delta(x)\cdot 1 = \delta(x).$$

But we would also have

$$\delta(x)\cdot x\cdot\mathrm{PV}(1/x) = (x\cdot\delta(x))\cdot\mathrm{PV}(1/x) = 0\cdot\mathrm{PV}(1/x) = 0.$$

This is a contradiction.

Problem 11.8: Fix $\varphi\in\mathcal{S}$. Set $\alpha = \int\varphi$, and define

$$\psi(x) = \int_{-\infty}^{x}\big(\varphi(z) - \alpha\,\omega(z)\big)\,dz. \tag{3}$$

Obviously, $\psi\in C^\infty$, and

$$\varphi(x) = \alpha\omega(x) + \psi'(x). \tag{4}$$

Moreover, we find that if $n\geq 1$, then

$$||\psi||_{n,k} = ||(1+|x|^2)^{k/2}\psi^{(n)}||_{\mathrm{u}}$$
$$= ||(1+|x|^2)^{k/2}(\varphi^{(n-1)} - \alpha\omega^{(n-1)})||_{\mathrm{u}} \leq ||\varphi||_{n-1,k} + |\alpha|\,||\omega||_{n-1,k}.$$

It remains to prove that for any k,

$$\sup_x (1+|x|^2)^{k/2}|\psi(x)| < \infty.$$

First consider $x \leq 0$. Then for any k, we have

$$\begin{aligned}
&\sup_{x\leq 0}(1+|x|^2)^{k/2}|\psi(x)| \\
&\qquad \leq \limsup_{x\leq 0}\Big[(1+|x|^2)^{k/2}\int_{-\infty}^{x}\frac{1}{(1+|y|^{(k+2)/2})}||\varphi||_{0,k+2}\,dy \\
&\qquad\qquad + |\alpha|(1+|x|^2)^{k/2}\int_{-\infty}^{x}\frac{1}{(1+|y|^{(k+2)/2})}||\omega||_{0,k+2}\,dy\Big] < \infty.
\end{aligned}$$

To prove the corresponding estimate for $x \geq 0$, we use that since

$$\underbrace{\int_{-\infty}^{x}\big(\varphi(z)-\alpha\,\omega(z)\big)\,dz}_{=\psi(x)} + \int_{x}^{\infty}\big(\varphi(z)-\alpha\,\omega(z)\big)\,dz = 0,$$

we can also express ψ as

$$\psi(x) = -\int_{x}^{\infty}\big(\varphi(z)-\alpha\,\omega(z)\big)\,dz.$$

Then proceed as in the bound for $x \leq 0$.

Problem 1:

$$\begin{aligned}
\langle D\,f,\varphi\rangle = -\langle f,\varphi'\rangle &= -\int_{-\infty}^{0}(-x)\varphi'(x)\,dx - \int_{0}^{\infty}x\varphi'(x)\,dx \\
&= \underbrace{[x\varphi(x)]_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0}\varphi(x)\,dx - \underbrace{[x\varphi(x)]_{0}^{\infty}}_{=0} + \int_{-\infty}^{0}\varphi(x)\,dx = \langle g,\varphi\rangle,
\end{aligned}$$

where

$$g(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

So $D\,f = g$. (Note that the value of $g(0)$ is irrelevant, any finite value can be assigned.) Furthermore,

$$\begin{aligned}
\langle D^2\,f,\varphi\rangle = \langle D\,g,\varphi\rangle = -\langle g,\varphi'\rangle &= \int_{-\infty}^{0}\varphi'(x)\,dx - \int_{0}^{\infty}\varphi'(x)\,dx \\
&= [\varphi(x)]_{-\infty}^{0} - [\varphi(x)]_{0}^{\infty} = \varphi(0) - (-\varphi(0)) = 2\varphi(0) = \langle 2\delta,\varphi\rangle,
\end{aligned}$$

so $D^2 f = 2\delta$.

Problem 2: Assume that f satisfies the given assumptions. We will prove that for any α and k, there exists a number C and a finite integer N such that

$$||f\,\varphi||_{\alpha,k} \leq C\sum_{|\beta|,l\leq N}||\varphi||_{\beta,l}.$$

This immediately proves both that $f\,\varphi \in \mathcal{S}$, and that $f\,\varphi_n \to f\,\varphi$ whenever $\varphi_n \to \varphi$ in $\mathcal{S}$.

Fix α and k. Then

$$\begin{aligned}
||f\,\varphi||_{\alpha,k} &= \sup_{x}(1+|x|^2)^{k/2}|\partial^{\alpha}(f(x)\varphi(x))| \\
&= \sup_{x}(1+|x|^2)^{k/2}|\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}(\partial^{\gamma}f(x))(\partial^{\beta}\varphi(x))|.
\end{aligned}$$

Now using that for each γ there exist finite numbers N_γ and C_γ such that

$$|\partial^\gamma f(x)| \leq C_\gamma (1+|x|^2)^{N_\gamma/2}$$

we obtain

$$||f\,\varphi||_{\alpha,k} \leq \sup_x (1+|x|^2)^{k/2} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!} C_\gamma (1+|x|^2)^{N_\gamma/2} |(\partial^\beta \varphi(x))|$$

$$= \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!} C_\gamma ||\varphi||_{\beta,k+N_\gamma}.$$

Problem 3: Define for $n = 1, 2, 3, \ldots,$ the functions

$$\chi_n(x) = \begin{cases} 1 & x \in \left[n - \frac{1}{4^n},\, n\right], \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$f(x) = \sum_{n=1}^{\infty} 2^n \chi_n(x).$$

Now (2) clearly holds for any k. To prove (3) note that for any given k, we have

$$\int_{-\infty}^{\infty} (1+|x|^2)^{k/2} |f(x)|\, dx = \sum_{n=1}^{\infty} \int_{n-4^{-n}}^{n} (1+|x|^2)^{k/2} |f(x)|\, dx$$

$$\leq \sum_{n=1}^{\infty} \int_{n-4^{-n}}^{n} (1+n^2)^{k/2} 2^n\, dx = \sum_{n=1}^{\infty} \frac{1}{2^n} (1+n^2)^{k/2} < \infty.$$