Solutions for homework set 8 — APPM5450, Spring 2013

Problem 11.1: For (b) prove that all the elements of the sub-base are convex just as you proved that balls in an NLS are convex. Then prove that any intersection of convex sets must be convex, which shows that the base must be convex. For (c), pick any $x, y \in X$. Since $(p_\alpha)_{\alpha \in A}$ separates points, there is a $\beta \in A$ such that $p_\beta(x-y)$ is non zero. Set $\varepsilon = p_\beta(x-y)/3$. Then the sets $\Omega_x = \{z \in X : p_\beta(x-z) < \varepsilon\}$ and $\Omega_y = \{z \in X : p_\beta(y-z) < \varepsilon\}$ are open disjoint neighborhoods of x and y, respectively.

Problem 11.2: *[Note: It is necessary to assume that $(p_n)_{n=1}^\infty$ separates points.]*

First prove that for any non-negative numbers a, b, and c such that $a \leq b + c$ we have

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

This is easily done by just multiplying both sides by $(1+a)(1+b)(1+c)$. This shows that if p is a semi-norm, then

$$\frac{p(x-y)}{1+p(x-y)} \leq \frac{p(x-z)}{1+p(x-z)} + \frac{p(z-y)}{1+p(z-y)}, \qquad \forall x, y, z \in X.$$

Then

$$d(x,y) = \sum_{n=1}^\infty \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)} \leq \sum_{n=1}^\infty \frac{1}{2^n} \left(\frac{p_n(x-z)}{1+p_n(x-z)} + \frac{p_n(z-y)}{1+p_n(z-y)} \right) = d(x,z) + d(z,y).$$

So d satisfies the triangle inequality.

We next need to prove that $d(x,y) = 0$ iff $x = y$. Suppose $x = y$. Then $p_n(x-y) = 0$ for all n, so $d(x,y) = 0$. Suppose next that $x \neq y$. Then for some n_0 we have $p_{n_0}(x-y) > 0$, and so $d(x,y) \geq 2^{-n_0} p_{n_0}(x-y)/(1+p_{n_0}(x-y)) > 0$.

It remains to prove that the metric $\mathcal{T}_d$ induced by d equals the topology $\mathcal{T}$ induced by the family of semi-norms.. Suppose $x_j \to x$ in $\mathcal{T}$. Fix $\varepsilon > 0$. Pick N such that $\sum_{n=N+1}^\infty 2^{-n} < \varepsilon$. Then

$$d(x, x_j) = \sum_{n=1}^\infty \frac{1}{2^n} \frac{p_n(x-x_j)}{1+p_n(x-x_j)} \leq \varepsilon + \sum_{n=1}^N \frac{1}{2^n} \frac{p_n(x-x_j)}{1+p_n(x-x_j)} \to \varepsilon, \qquad \text{as } j \to \infty.$$

Since ε was arbitrary, $d(x, x_j) \to 0$ and so $x_j \to x$ in $\mathcal{T}_d$. Now suppose x_j does not converge to x in $\mathcal{T}$. Then for some seminorm p_{n_0}, there exists a $\varepsilon > 0$ and a subsequence $(x_{j_k})_{k=1}^\infty$ such that $p_{n_0}(x_{j_k} - x) > \varepsilon$. Then

$$d(x_{j_k}, x) > 2^{-n_0} p_{n_0}(x_{j_k} - x)/(1 + p_{n_0}(x-y)) > 2^{-n_0} \varepsilon/(1+\varepsilon)$$

so x_j does not converge to x in $\mathcal{T}_d$ either.

Problem 11.6: We find that

$$\begin{aligned} \langle D(\log|x|)\, \varphi \rangle = -\langle \log|x|\, \varphi' \rangle = -\int_{\mathbb{R}} \log|x|\, \varphi'(x)\, dx \\ = -\lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{-\varepsilon} \log(-x) \varphi'(x)\, dx + \int_\varepsilon^\infty \log(x) \varphi'(x)\, dx \right\}. \end{aligned}$$

Partial integrations yield

$$- \langle \log|x|\, \varphi' \rangle = -\lim_{\varepsilon\to 0}\Big\{ [\log(-x)\varphi(x)]_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon} \frac{1}{-x}\varphi(x)\,dx + \\ [\log(x)\varphi(x)]_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{1}{x}\varphi(x)\,dx \Big\} \\ = \langle \mathrm{PV}(1/x),\, \varphi\rangle + \lim_{\varepsilon\to 0}\big\{\log(\varepsilon)\big(\varphi(\varepsilon)-\varphi(-\varepsilon)\big)\big\}.$$

Since

$$\big|\varphi(\varepsilon)-\varphi(-\varepsilon)\big| = \left|\int_{-\varepsilon}^{\varepsilon}\varphi'(x)\,dx\right| \leq 2\varepsilon||\varphi||_{1,0}$$

and $\lim_{\varepsilon\to 0}\{\varepsilon\log\varepsilon\} = 0$, we find that $\lim_{\varepsilon\to\ 0}\big\{\log(\varepsilon)\big(\varphi(\varepsilon)-\varphi(-\varepsilon)\big)\big\} = 0$.

Problem 11.7: First prove that $x\cdot\delta(x) = 0$ and that $x\cdot\mathrm{PV}(1/x) = 1$ (using the regular rules for the product between a polynomial and a Schwartz function). Suppose that $\cdot$ is distributive and can pair any two distributions. Then on the one hand we would have

$$\delta(x)\cdot x\cdot \mathrm{PV}(1/x) = \delta(x)\cdot(x\cdot\mathrm{PV}(1/x)) = \delta(x)\cdot 1 = \delta(x).$$

But we would also have

$$\delta(x)\cdot x\cdot\mathrm{PV}(1/x) = (x\cdot\delta(x))\cdot\mathrm{PV}(1/x) = 0\cdot\mathrm{PV}(1/x) = 0.$$

This is a contradiction.

Problem 11.8: Fix $\varphi\in\mathcal{S}$. Set $\alpha = \int\varphi$, and define

$$\psi(x) = \int_{-\infty}^{x}\big(\varphi(z) - \alpha\,\omega(z)\big)\,dz. \tag{1}$$

Obviously, $\psi\in C^\infty$, and

$$\varphi(x) = \alpha\omega(x) + \psi'(x). \tag{2}$$

Moreover, we find that if $n\geq 1$, then

$$||\psi||_{n,k} = ||(1+|x|^2)^{k/2}\psi^{(n)}||_{\mathrm{u}} \\ = ||(1+|x|^2)^{k/2}(\varphi^{(n-1)} - \alpha\omega^{(n-1)})||_{\mathrm{u}} \leq ||\varphi||_{n-1,k} + |\alpha|\,||\omega||_{n-1,k}.$$

It remains to prove that for any k,

$$\sup_x (1+|x|^2)^{k/2}|\psi(x)| < \infty.$$

First consider $x\leq 0$. Then for any k, we have

$$\sup_{x\leq 0}(1+|x|^2)^{k/2}|\psi(x)| \\ \leq \limsup_{x\leq 0}\Big[(1+|x|^2)^{k/2}\int_{-\infty}^{x}\frac{1}{(1+|y|^{(k+2)/2})}||\varphi||_{0,k+2}\,dy \\ + |\alpha|(1+|x|^2)^{k/2}\int_{-\infty}^{x}\frac{1}{(1+|y|^{(k+2)/2})}||\omega||_{0,k+2}\,dy\Big] < \infty.$$

To prove the corresponding estimate for $x \geq 0$, we use that since

$$\underbrace{\int_{-\infty}^{x} \big(\varphi(z) - \alpha\,\omega(z)\big)\,dz}_{=\psi(x)} + \int_{x}^{\infty} \big(\varphi(z) - \alpha\,\omega(z)\big)\,dz = 0,$$

we can also express ψ as

$$\psi(x) = -\int_{x}^{\infty} \big(\varphi(z) - \alpha\,\omega(z)\big)\,dz.$$

Then proceed as in the bound for $x \leq 0$.

Problem 11.10:

(a) Fix h. We need to prove that if $\varphi_n \to \varphi$ in $\mathcal{S}$, then $\tau_h\varphi_n \to \tau_h\varphi$ in $\mathcal{S}$.

Suppose that $\varphi_n \to \varphi$ in $\mathcal{S}$. Fix α, k. Then

$$p_{\alpha,k}(\tau_h\varphi) = \sup_y \big(1+|y|^2\big)^{k/2}\,|\partial^\alpha\varphi(y-h)| = \sup_x \big(1+|x+h|^2\big)^{k/2}\,|\partial^\alpha\varphi(x)|$$
$$\leq \sup_x \big(1+2\,|x|^2+2\,|h|^2\big)^{k/2}\,|\partial^\alpha\varphi(x)|.$$

Now use that $1+2\,|h|^2+2\,|x|^2 \leq 2(1+|h|^2)(1+|x|^2)$ to get

$$p_{\alpha,k}(\tau_h\varphi) \leq \sup_x \big(2(1+|x|^2)(1+|h|^2)\big)^{k/2}\,|\partial^\alpha\varphi(x)|$$
$$= 2^{k/2}(1+|h|^2)^{k/2}\sup_x \big(1+|x|^2\big)^{k/2}\,|\partial^\alpha\varphi(x)| = 2^{k/2}(1+|h|^2)^{k/2}p_{\alpha,k}(\varphi).$$

Now we can easily prove that $\tau_h\varphi_n \to \varphi$ in $\mathcal{S}$. We have

$$p_{\alpha,k}(\tau_h\varphi_n - \tau_h\varphi) = p_{\alpha,k}(\tau_h(\varphi_n-\varphi)) \leq 2^{k/2}(1+|h|^2)^{k/2}p_{\alpha,k}(\varphi_n-\varphi).$$

Since $p_{\alpha,k}(\varphi_n-\varphi)\to 0$ by assumption, it follows that $p_{\alpha,k}(\tau_h\varphi_n - \tau_h\varphi)\to 0$, and since α, k were arbitrary, it follows that $\tau_h\varphi_n \to \tau_h\varphi$ in $\mathcal{S}$.

(b) Fix $\varphi \in \mathcal{S}$. We will first prove that $\tau_h\varphi \to \varphi$ in $\mathcal{S}$ when $h \to 0$ in $\mathbb{R}^d$.

Fix α and k. We need to prove that $p_{\alpha,k}(\tau_h\varphi - \varphi)\to 0$ as $h\to 0$. To this end, fix a real number R (to be determined) and observe that

$$p_{\alpha,k}(\tau_h\varphi-\varphi) = \sup_x (1+|x|^2)^{k/2}|\partial^\alpha(\varphi(x-h)-\varphi(x))|$$
$$\leq \underbrace{\sup_{|x|>R}(1+|x|^2)^{k/2}|\partial^\alpha\varphi(x)|}_{=:c_1} + \underbrace{\sup_{|x|>R}(1+|x|^2)^{k/2}|\partial^\alpha\varphi(x-h)|}_{=:c_2} + \underbrace{\sup_{|x|\leq R}(1+|x|^2)^{k/2}|\partial^\alpha(\varphi(x-h)-\varphi(x))|}_{=:c_3}.$$

Fix $\varepsilon > 0$. Since $\partial^\alpha\varphi$ decays faster than any polynomial, we can find a number R such that $c_1 + c_2 \leq \varepsilon$. To handle c_3, we observe that

$$|\partial^\alpha(\varphi(x-h)-\varphi(x))| = \left|\int_0^1 h\cdot\nabla(\partial^\alpha\varphi(x-t\,h))\,dt\right| \leq |h|_1 \sup_{|\beta|=|\alpha|+1}||\partial^\beta\varphi||_{\mathrm{u}} = |h|_1 \sup_{|\beta|=|\alpha|+1}||\varphi||_{0,\beta}.$$

It follows that

$$c_3 \leq (1+R^2)^{k/2}\,|h|_1 \sup_{|\beta|=|\alpha|+1}||\varphi||_{0,\beta}.$$

Taking the limit as $h\to 0$, we find that

$$\limsup_{h\to 0} p_{\alpha,k}(\tau_h\varphi-\varphi) \leq \limsup_{h\to 0}(c_1+c_2+c_3) \leq \limsup_{h\to 0}(\varepsilon + (1+R^2)^{k/2}\,|h|_1 \sup_{|\beta|=|\alpha|+1}||\varphi||_{0,\beta}) = \varepsilon.$$

Since ε was arbitrary, it follows that

$$\lim_{h\to 0} p_{\alpha,k}(\tau_h\varphi-\varphi) = 0$$

which completes the first part of the proof.

It remains to prove that for any $\psi \in \mathcal{S}$ it is the case that $\tau_{h'}\psi \to \tau_h\psi$ in $\mathcal{S}$ when $h' \to h$ in $\mathbb{R}^d$. But this is easy due to group property. Simply set $\varphi = \tau_h\psi$ and $g = h' - h$. Then $g \to 0$, so $\tau_g\varphi \to \varphi$ in $\mathcal{S}$. But $\tau_g\varphi = \tau_{h'-h}\,\tau_h\,\psi = \tau_{h'}\,\psi$ so we are done.

Problem 1:

$$\langle D\,f, \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{-\infty}^{0}(-x)\varphi'(x)\,dx - \int_0^{\infty} x\varphi'(x)\,dx$$
$$= \underbrace{[x\varphi(x)]_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0}\varphi(x)\,dx - \underbrace{[x\varphi(x)]_0^{\infty}}_{=0} + \int_{-\infty}^{0}\varphi(x)\,dx = \langle g, \varphi\rangle,$$

where

$$g(x) = \begin{cases} -1 & x \le 0 \\ 1 & x > 0. \end{cases}$$

So $D\,f = g$. (Note that the value of $g(0)$ is irrelevant, any finite value can be assigned.) Furthermore,

$$\langle D^2\,f, \varphi\rangle = \langle D\,g, \varphi\rangle = -\langle g, \varphi'\rangle = \int_{-\infty}^{0}\varphi'(x)\,dx - \int_0^{\infty}\varphi'(x)\,dx$$
$$= [\varphi(x)]_{-\infty}^{0} - [\varphi(x)]_0^{\infty} = \varphi(0) - (-\varphi(0)) = 2\varphi(0) = \langle 2\delta, \varphi\rangle,$$

so $D^2 f = 2\delta$.

Problem 2: Assume that f satisfies the given assumptions. We will prove that for any α and k, there exists a number C and a finite integer N such that

$$||f\,\varphi||_{\alpha,k} \le C \sum_{|\beta|,l \le N} ||\varphi||_{\beta,l}.$$

This immediately proves both that $f\,\varphi \in \mathcal{S}$, and that $f\,\varphi_n \to f\,\varphi$ whenever $\varphi_n \to \varphi$ in $\mathcal{S}$.

Fix α and k. Then

$$||f\,\varphi||_{\alpha,k} = \sup_x (1+|x|^2)^{k/2}|\partial^\alpha(f(x)\varphi(x))|$$
$$= \sup_x (1+|x|^2)^{k/2}|\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}(\partial^\gamma f(x))(\partial^\beta\varphi(x))|.$$

Now using that for each γ there exist finite numbers N_γ and C_γ such that

$$|\partial^\gamma f(x)| \le C_\gamma(1+|x|^2)^{N_\gamma/2}$$

we obtain

$$||f\,\varphi||_{\alpha,k} \le \sup_x (1+|x|^2)^{k/2}\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}C_\gamma(1+|x|^2)^{N_\gamma/2}|(\partial^\beta\varphi(x))|$$
$$= \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}C_\gamma||\varphi||_{\beta,k+N_\gamma}.$$

Problem 3: Define for $n = 1, 2, 3, \ldots,$ the functions

$$\chi_n(x) = \begin{cases} 1 & x \in \left[n - \frac{1}{4^n},\, n\right], \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$f(x) = \sum_{n=1}^{\infty} 2^n \, \chi_n(x).$$

Now (2) clearly holds for any k. To prove (3) note that for any given k, we have

$$\int_{-\infty}^{\infty} (1 + |x|^2)^{k/2} \, |f(x)| \, dx = \sum_{n=1}^{\infty} \int_{n-4^{-n}}^{n} (1 + |x|^2)^{k/2} |f(x)| \, dx$$
$$\leq \sum_{n=1}^{\infty} \int_{n-4^{-n}}^{n} (1 + n^2)^{k/2} 2^n \, dx = \sum_{n=1}^{\infty} \frac{1}{2^n} (1 + n^2)^{k/2} < \infty.$$