APPM5450 — Applied Analysis: Final exam

7:30 – 9:50, May 9, 2013. Closed books.

Please motivate your answers unless the problem explicitly states otherwise.

Problem 1: (12p) No motivation required for these problems.

(a) (3p) Let $n \in \mathbb{Z}$ and define $f_n \in \mathcal{S}^*(\mathbb{R})$ via $f_n(x) = \sin(nx)$. What is $\hat{f}_n$?

(b) (3p) State for which $p \in [1, \infty]$, if any, the unit ball in $L^p(\mathbb{R})$ is weakly compact.

(c) (3p) Set $H = L^2(\mathbb{R})$ and define $T \in \mathcal{B}(H)$ via $[Tu](x) = u(-x)$. What is $\sigma(T)$?

(d) (3p) Let H be a Hilbert space. State the definition of a *unitary* operator on H.

Problem 2: (13p) Let H be a Hilbert space, and let A denote a bounded linear operator on H.

(a) (3p) State the definition of the *resolvent set* $\rho(A)$ of A.

(b) (10p) Prove that the resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$.

Problem 3: (16p) Define for $\alpha, \beta \in (0, \infty)$ and for $n = 1, 2, 3, \dots$ functionals A_n, $B_n \in \mathcal{S}^*(\mathbb{R})$ via

$$A_n(\varphi) = \sum_{j=1}^{n} \alpha^j \, \varphi(j), \qquad \text{and} \qquad B_n(\varphi) = \sum_{j=1}^{n} j^\beta \, \varphi(j).$$

(a) (8p) For which $\alpha \in (0, \infty)$ does the sequence $(A_n)_{n=1}^\infty$ converge in $\mathcal{S}^*(\mathbb{R})$?

(b) (8p) For which $\beta \in (0, \infty)$ does the sequence $(B_n)_{n=1}^\infty$ converge in $\mathcal{S}^*(\mathbb{R})$?

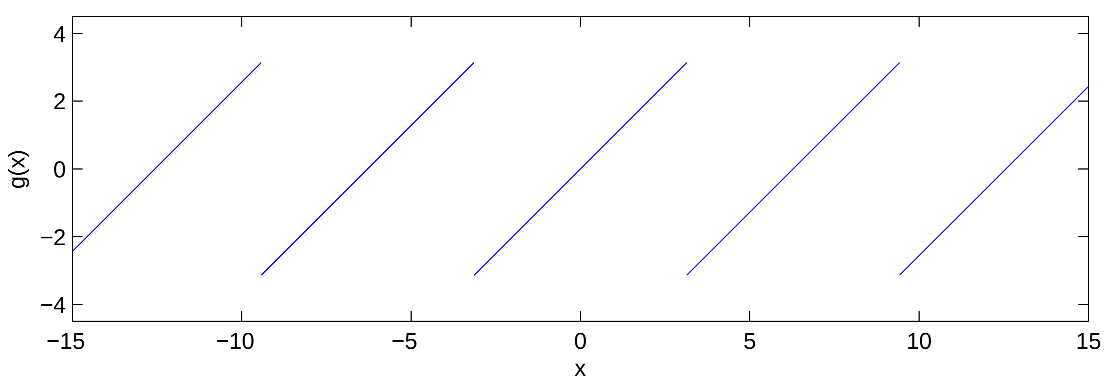
Problem 4: (23p) Let $\mathbb{T}$ denote the unit circle as usual, and define a function $f \in L^2(\mathbb{T})$ via $f(x) = x$, where $\mathbb{T}$ is parameterized using $x \in [-\pi, \pi)$.

(a) (5p) What are the Fourier coefficients of f?

(b) (5p) For which $s \in [0, \infty)$ is it the case that $f \in H^s(\mathbb{T})$?

(c) (5p) Use your result in (a) to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

(d) (5p) Let g denote the real-valued function obtained via periodic continuation of f to a 2π periodic function on $\mathbb{R}$ (see figure below). Prove that $g \in \mathcal{S}^*(\mathbb{R})$.



(e) (3p) What is the Fourier transform of the function $g \in \mathcal{S}^*(\mathbb{R})$ defined in (d)? No motivation required for this part. (Hint: Problem 1(a) may be useful.)

Problem 5: (18p) Set $I = (0, 1)$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of Lebesgue integrable real valued functions on the interval $I = (0, 1)$ such that for every $x \in I$,

$$\lim_{n\to\infty} f_n(x) = x.$$

Consider for $n = 1, 2, 3, \ldots$ the three sequences

$$a_n = \int_0^1 f_n(x)\,dx$$

$$b_n = \int_0^1 \frac{f_n(x)}{1 + (f_n(x))^2}\,dx$$

$$c_n = \int_0^1 \left|\sum_{j=1}^{n} f_j(x)\right| dx.$$

Which of the sequences must necessarily converge as $n \to \infty$? Is it for any of the convergent sequences possible to say what the limit is? Motivate your answers.

Problem 6: (18p) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^2(\mathbb{R})$ that converges pointwise to a function f. In other words,

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for all } x \in \mathbb{R}.$$

Suppose further that all f_n satisfy

$$|f_n(x)| \le 2|f(x)|, \qquad \text{for all } x \in \mathbb{R}.$$

For each of the three sets of conditions on f given below, specify for which $r \in [1, \infty)$ it is necessarily the case that

$$\lim_{n\to\infty} ||f - f_n||_{L^r(\mathbb{R})} = 0.$$

(a) (6p) $f \in L^2(\mathbb{R})$, and for $|x| \ge 2$, it is the case that $f(x) = 0$.

(b) (6p) $f \in L^2(\mathbb{R})$ and $|f(x)| \le 2$ for all $x \in \mathbb{R}$.

(c) (6p) $f \in L^2(\mathbb{R})$ and $f \in L^3(\mathbb{R})$.