SELF-ADJOINT OPERATORS

Thm Let H be a H.S., and let $A \in \mathcal{B}(H)$ be S-A. Then:

(a) $\sigma_p(A) \subseteq \mathbb{R}$

(b) If $\lambda, \mu \in \sigma_p(A)$ and $\lambda \neq \mu$, then $\ker(A-\lambda I) \perp \ker(A-\mu I)$

Proof (a) Suppose $Ax = \lambda x$ and $x \neq 0$. Then

$$\lambda \|x\|^2 = \lambda (x,x) = (x, \lambda x) = (x, Ax) = (Ax, x) = (\lambda x, x) = \bar{\lambda} \|x\|^2$$

(b) Suppose $Ax = \lambda x$, $Ay = \mu y$, and $\lambda \neq \mu$. Then

$$\lambda (x,y) \underset{\substack{\uparrow \\ \lambda \in \mathbb{R}}}{=} (\lambda x, y) = (Ax, y) = (x, Ay) = (x, \mu y) = \mu (x,y)$$

so $\underbrace{(\lambda - \mu)}_{\neq 0}(x,y) = 0$

Thm Let H be a H.S, and let $A \in \mathcal{B}(H)$ be S-A. Then

(a) $\sigma(A) \subseteq \mathbb{R}$

(b) $\sigma_r(A) = \emptyset$

Lemma Let H be a H.S., and let $A \in \mathcal{B}(H)$.
Then if $\lambda \in \sigma_r(A)$, then $\bar{\lambda} \in \sigma_p(A^*)$.

Proof of lemma Suppose $\lambda \in \sigma_r(A)$.

Then $\overline{\operatorname{ran}(A-\lambda I)} \neq H$ so $\exists\ x \in \operatorname{ran}(A-\lambda I)^\perp$ s.t. $x \neq 0$.

Now $\operatorname{ran}(A-\lambda I)^\perp = \ker(A^* - \bar{\lambda} I)$ so $A^* x = \bar{\lambda} x$.

Proof of thm

(a) Suppose $\lambda = a + ib$ with $b \neq 0$. Then

$$\|(A-\lambda I)x\|^2 = \|(A-aI)x - ibx\|^2 =$$

$$= \underbrace{\|(A-aI)x\|^2}_{\geq 0} - 2\mathrm{Re}\Big[\underbrace{((A-aI)x, ibx)}_{= i\underbrace{((A-aI)x, bx)}_{\in \mathbb{R}}}\Big] + \underbrace{\|ibx\|^2}_{= b^2\|x\|^2} \geq b^2\|x\|^2.$$

Since $A - \lambda I$ is coercive, $\ker(A-\lambda I) = \{0\}$ so $\lambda \notin \sigma_p(A)$.
Moreover ~~Ran(A-λI) is closed~~ $A - \lambda I$ has closed range, so $\lambda \notin \sigma_c(A)$.
Finally, if λ were to be in $\sigma_r(A)$, then $\bar{\lambda} = a - ib \in \sigma_p(A^*) = \sigma_p(A)$
which is impossible since $\sigma_p(A) \subseteq \mathbb{R}$.

(b) Suppose that $\lambda \in \sigma_r(A)$. Then $\bar{\lambda} \in \sigma_p(A^*) = \sigma_p(A)$.
Therefore $\lambda = \bar{\lambda}$ and $\lambda \in \sigma_r(A) \cap \sigma_p(A) = \emptyset$.

Def^n Let H be a H.S, and let $A \in \mathcal{B}(H)$.
For $\lambda \in \sigma_p(A)$, define the <u>multiplicity</u> of λ
as $\dim(\ker(A - \lambda I))$

Note In a H.S., the multiplicity may in general be infinite.
As an example, consider $A = I$ and $\lambda = 1$.
Then $\ker(A - \lambda I) = \ker(I - I) = \ker(0) = H$.

Thm Let H be a H.S., and let $A \in \mathcal{B}(H)$ be compact & S.A. Then

(a) If $\lambda \in \sigma_p(A)$ and $\lambda \neq 0$, then λ has finite multiplicity

(b) If $\sigma_p(A)$ is infinite, then 0 is an accumulation point of $\sigma_p(A)$, and there are no other accumulation points.

Note: The thm implies that the non-zero evals of A can be ordered so that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ and that $|\lambda_n| \to 0$ as $n \to \infty$.

Proof: (a) By contradiction.

Suppose $\lambda \in \sigma_p(A)$, $\lambda \neq 0$, and $\dim(\ker(A - \lambda I)) = \infty$.

Then $\exists$ an ON seq $(e_n)_{n=1}^{\infty}$ s.t. $Ae_n = \lambda e_n$.

$e_n \rightharpoonup 0$, and A is compact so $Ae_n \to 0$.

But this is impossible since $\|Ae_n - 0\| = \|\lambda e_n - 0\| = |\lambda|$

(b) Suppose that $\sigma_p(A)$ is infinite.

Since $\sigma_p(A)$ is bdd ($|\lambda| \leq \|A\|$ when $\lambda \in \sigma(A)$) there must be at least one accumulation point λ.

We will prove that if $\lambda \neq 0$, then λ cannot be an acc. point.

Suppose $\lambda \neq 0$. Then we can pick $\lambda_n \in \sigma_p(A)$ s.t.

$\lambda_n \to \lambda$, $|\lambda_n| \geq \frac{|\lambda|}{2}$ $\forall n$, and $\lambda_n \neq \lambda_m$ when $n \neq m$.

Let e_n be s.t. $Ae_n = \lambda_n e_n$.

Set $f_n = \frac{1}{\lambda_n} e_n$. Then $\|f_n\| = \frac{1}{|\lambda_n|} \leq \frac{2}{|\lambda|}$ so we can pick a convergent subseq $f_{n_j} \rightharpoonup f$.

Since A is compact, $Af_{n_j} \to Af$.

This is impossible since $Af_{n_j} = A\frac{1}{\lambda_{n_j}} e_{n_j} = e_{n_j}$.

Alternative end:
Set $\Omega = \{\frac{1}{\lambda_n} e_n\}_{n=1}^{\infty}$
Ω bdd so $A\Omega$ precompact
$A\Omega = \{e_n\}_{n=1}^{\infty}$
which is impossible

Lemma Let H be H.S., and let $A \in \mathcal{B}(H)$ be S-A and compact.

Then either $\|A\|$, or $-\|A\|$, or both, belong to $\sigma_p(A)$

Proof Recall that $\|A\| = \sup_{\|u\|=1} |(Au, u)|$

Therefore, there is a seq $(u_n)_{n=1}^{\infty}$ s.t. $\|u_n\| = 1$, and

$(Au_n, u_n) \to \lambda$ where $\lambda = \pm\|A\|$.

(u_n) bdd $\Rightarrow \exists$ subseq $(u_{n_j})_{j=1}^{\infty}$ s.t. $u_{n_j} \rightharpoonup u$.

Since A is compact $Au_{n_j} \to Au =: v$.

We will prove that $Av = \lambda v$:

$$\|(A - \lambda I)v\|^2 = \lim_{j\to\infty} \|(A - \lambda I)Au_{n_j}\|^2 \le \|A\|^2 \limsup_{j\to\infty} \|(A - \lambda I)u_{n_j}\|^2 =$$

$$= \|A\|^2 \limsup_{j\to\infty} \Big[\underbrace{\|Au_{n_j}\|^2}_{\le \lambda^2} - 2\lambda \underbrace{(Au_{n_j}, u_{n_j})}_{\to \lambda} + \lambda^2 \underbrace{\|u_{n_j}\|^2}_{=1} \Big] = 0$$

It only remains to prove that $v \ne 0$.

Suppose $v = 0$. Then $\|Au_{n_j}\| \to 0$, whence

$$\|A\| = |\lambda| = \lim_{j\to\infty} |(Au_{n_j}, u_{n_j})| \le \limsup_{j\to\infty} \|Au_{n_j}\| = 0$$

Invariant subspaces

Let H be a H.S, and suppose $H = M \oplus N$ where $M = N^{\perp}$. ~~The P~~ Let P & Q denote orthog proj^ns onto M & N.

Write $x \sim \begin{bmatrix} Px \\ Qx \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Now let $A \in \mathcal{B}(H)$. We have, for $y = Ax$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} PAx \\ QAx \end{bmatrix} \overset{P+Q=I}{=} \begin{bmatrix} PAPx + PAQx \\ QAPx + QAQx \end{bmatrix} = \begin{bmatrix} PAP & PAQ \\ QAP & QAQ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now suppose that M, N are <u>invariant subspaces</u> of A which means that $AM \subseteq M$ (i.e. $Ax \in M$, whenever $x \in M$)
$AN \subseteq N$ (i.e. $Ax \in N$, whenever $x \in N$).

Then $PAQ = 0$ & $QAP = 0$ so

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} PAP & 0 \\ 0 & QAQ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ ← the operator is <u>block diagonal</u>.

Now, if A is S-A, and $AM \subseteq M$, then $AM^{\perp} \subseteq M^{\perp}$ automatically!

<u>Lemma</u> Suppose H is a H.S. and that $A \in \mathcal{B}(H)$ is S-A.
Then if M is an invariant subspace of A, so is $M^{\perp}$.

<u>Proof</u> Suppose $x \in M^{\perp}$.
$\forall y \in M$: $(Ax, y) = (x, Ay) = 0$ since $x \in M^{\perp}$ and $Ay \in M$.

<u>Note</u> The lemma is not true for general ops:

$H = \mathbb{C}^2$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $M = \mathrm{span}(e_1)$ ← invariant
$M^{\perp} = \mathrm{span}(e_2)$ ← not invariant.

Suppose A is S-A, and $Av = \lambda v$ for some $v \neq 0$.

Then $M = \text{span}(v)$ is an invariant subspace.

$(x \in M \Rightarrow x = \alpha v \Rightarrow Ax = \alpha Av = \alpha \lambda v \in M)$

Let P denote proj^n onto M.

Then $APx = \lambda Px$ so $AP = \lambda P$.

Example $H = \mathbb{C}^n$ $A \in \mathcal{B}(H)$ is S-A.

Then A has an ON-basis $(e_j)_{j=1}^n$ s.t. $Ae_j = \lambda_j e_j$.

Let P_j denote orthog proj^n onto span(e_j). $\rightarrow I = \sum_{j=1}^n P_j$

Then $AP_j x = \lambda_j P_j x$ and

$$A = A \sum_{j=1}^n P_j = \sum_{j=1}^n \lambda_j P_j = \sum_{j=1}^n \lambda_j e_j e_j^*$$

Thm Let H be a H.S., and let $A \in \mathcal{B}(H)$ be compact and S-A.

Then there is an ON-seq $(e_n)_{n=1}^N$ (N may be infinite) s.t.

* $Ae_n = \lambda_n e_n$
* $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$
* If $N = \infty$, then $|\lambda_n| \to 0$
* $A = \sum_{n=1}^N \lambda_n P_n$ where $P_n x = e_n (e_n, x)$ and the sum converges in norm if $N = \infty$.

Moreover, if $\ker(A) = \{0\}$, then $\{e_n\}_{n=1}^N$ is an ON-basis for H.

If $\ker(A) \neq \{0\}$, and if $\{f_m\}_{m=1}^M$ is an ON-basis for $\ker(A)$, then $(e_n)_{n=1}^N \cup (f_m)_{m=1}^M$ is an ON-basis for H.

Proof First we construct subspaces (M_n) and (N_n), and operators (A_n) via the following procedure:

Step 1 Set $N_1 = H$, ~~and~~ $M_1 = \{0\}$, and $A_1 = A$.
$\exists\, \lambda_1$ and e_1 s.t. $Ae_1 = \lambda_1 e_1$, $\|e_1\| = 1$, and $|\lambda_1| = \|A_1\|$.
Set P_1 = proj^n onto span(e_1).

Step 2 Set $M_2 = \text{span}(e_1)$ and $N_2 = M_2^{\perp}$,
and let A_2 denote the restriction of A to N_2, $A_2 = A_1 - \lambda_1 P_1$
$\exists\, \lambda_2$ and e_2 s.t. $Ae_2 = \lambda_2 e_2$, $\|e_2\| = 1$, and $|\lambda_2| = \|A_2\|$
Set P_2 = orthog proj^n onto span(e_2).

$\vdots$

Step n Set $M_n = \text{span}(e_1, e_2, \ldots, e_{n-1})$ and $N_n = M_n^{\perp}$,
and let A_n denote the restriction of A to N_n, $A_n = A - \sum_{j=1}^{n-1} \lambda_j P_j$
$\exists\, \lambda_n$ and e_n s.t. $Ae_n = \lambda_n e_n$, $\|e_n\| = 1$, and $|\lambda_n| = \|A_n\|$
Set P_n = orthog proj^n onto span(e_n).

Note that at the n'th step, $A = \sum_{j=1}^{n} \lambda_j P_j + A_{n+1}$

Proof cont'd

The process may end in two ways:

Case 1 For some n, $A_{n+1}=0$.

In this case A has finite rank, $A=\sum_{j=1}^{n}\lambda_j P_j$

$H=\mathrm{span}(e_1,e_2,\dots,e_n)\oplus\ker(A)$

Let $(f_m)_{m=1}^{M}$ be an ON-basis for $\ker(A)$.

Case 2 $A_n\neq 0\ \forall n$.

Then $\|A - \sum_{j=1}^{n}\lambda_j P_j\| = \|A_{n+1}\| = |\lambda_{n+1}| \to 0$ as $n\to\infty$.

So $A=\sum_{n=1}^{\infty}\lambda_n P_n$

However, (e_n) is not necessarily a basis.

If $\overline{\mathrm{span}(e_n)}=H$, then it is, and we are done.

If not, then suppose $x\in \mathrm{span}(e_n)^{\perp}$ and $x\neq 0$.

Then $x\in N_n\ \forall n$ so

$\|Ax\|=\|A_n x\|\leq \|A_n\|\ \|x\|\to 0$ as $n\to\infty$

So $x\in\ker(A)$.

Thus $H=\overline{\mathrm{span}(e_n)}\oplus\ker(A)$.

Review of compact sets & compact operators on H.S.

Recall that if Ω is a subset of a H.S., then TFAE:

(a) Ω is compact

(b) Every open cover of Ω has a finite subcover

(c) Every sequence in Ω has a convergent subseq

(d) Ω is closed and totally bdd (i.e. $\forall \varepsilon>0, \exists (x_j)_{j=1}^{J}$ s.t. $\Omega \subseteq \bigcup_{j=1}^{J} B_\varepsilon(x_j)$)

Recall that if $A \in \mathcal{B}(H)$, then TFAE:

(1) A is compact

(2) For any bdd set Ω, $A\Omega$ is precompact

(3) If (x_n) is a bdd seq, then (Ax_n) has a convergent subseq

(4) If $x_n \to x$ weakly, then $Ax_n \to Ax$ in norm

(5) For any $\varepsilon>0$, $\exists$ a finite rank operator A_N s.t. $\|A-A_N\|<\varepsilon$

We will next show that in a H.S, we can add a property (e) to the list above that is the set equivalent of condition (5).

Proof that (3) $\Leftrightarrow$ (4):

(3) $\Rightarrow$ (4) Assume that ~~(3) holds and that~~ $x_n \to x$. but that $Ax_n \not\to Ax$.
Then $\forall y$: $(Ax - Ax_n, y) = (x - x_n, A^*y) \to 0$ so $Ax_n \to Ax$.
~~Then~~ Moreover $\exists$ $\varepsilon>0$ & $(n_j)_{j=1}^{\infty}$ s.t. $\|Ax_{n_j} - Ax\| > \varepsilon$ $\forall j$.
~~But then~~ Note that (x_{n_j}) is a bdd seq, but no subseq of (Ax_{n_j}) may converge strongly since $Ax_{n_j} \to Ax$ but $\|Ax_{n_j} - Ax\| > \varepsilon$
Thus (3) cannot hold.

(4) $\Rightarrow$ (3) Assume that (4) holds, and that (x_n) is a bdd seq.
Banach-Alaoglu $\Rightarrow \exists (n_j)$ s.t. $x_{n_j} \to x$ for some x.
Then (4) $\Rightarrow$ $Ax_{n_j} \to Ax$ which proves (3).

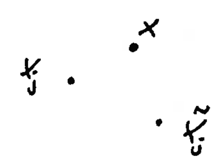
<u>Thm</u> Let Ω be a subset of an inf-dim. seperable H.S. H.

(a) If Ω is pre-compact, and $(\varphi_n)_{n=1}^{\infty}$ is an ON-basis for H then for any $\varepsilon>0$, $\exists N$ s.t. $\|(I-P_N)x\|<\varepsilon$ $\forall x\in\Omega$, where P_N = orthog proj^n onto $\mathrm{span}(\varphi_1,\dots,\varphi_N)$.

(b) If Ω is a bdd set, and $(\varphi_n)_{n=1}^{\infty}$ is an ON-basis for H, such that $\forall\varepsilon>0$, $\exists N$ s.t. $\|(I-P_N)x\|<\varepsilon$ $\forall x\in\Omega$, where P_N is as in (a), then Ω is pre-compact.

<u>Note</u>: As an implication of (a), if Ω is pre-compact, and $(\varphi_n)_{n=1}^{\infty}$ is any ON-set in H, then for any $\varepsilon>0$, $\exists N$ s.t. $\sum_{n=N+1}^{\infty}|(\varphi_n,x)|^2<\varepsilon^2$.
To see this simply apply the thm to the proj^n of H onto $\mathrm{span}(\varphi_n)_{n=1}^{\infty}$.

<u>Proof</u> (a) Suppose Ω is precompact and let $(\varphi_n)_{n=1}^{\infty}$ be an ON-basis for H.
Given $\varepsilon>0$, pick $(x_j)_{j=1}^{J}$ s.t. $\Omega\subseteq\bigcup_{j=1}^{J}B_{\varepsilon/2}(x_j)$.
For each j, pick N_j s.t. $\|(I-P_{N_j})x_j\|<\frac{\varepsilon}{2}$ (possible since $\|(I-P_N)x_j\|\to 0$ as $N\to\infty$)
Set $\tilde{x}_j=P_{N_j}x_j$.
Set $N=\max_{1\le j\le J}N_j$.
Then given $x\in\Omega$, pick x_j s.t. $x\in B_{\varepsilon/2}(x_j)$



then $\|(I-P_N)x\|=\|(I-P_N)(\tilde{x}_j+x-\tilde{x}_j)\|=\underbrace{\|(I-P_N)\tilde{x}_j\|}_{=0}+\|(I-P_N)(x-\tilde{x}_j)\|\le$
$\le\|x-\tilde{x}_j\|<\|x-x_j\|+\|x_j-\tilde{x}_j\|<\varepsilon$.

(b) Suppose that the conditions given in (b) hold.
Set $C=\sup\{\|x\|: x\in\Omega\}$.
Given $\varepsilon>0$, pick N s.t. $\|(I-P_N)x\|<\varepsilon/2$ $\forall x\in\Omega$.
Set $R_N=\{x\in\mathrm{span}(\varphi_1,\dots,\varphi_N): \|x\|\le C\}$
R_N is compact $\Rightarrow\exists(x_j)_{j=1}^{J}$ s.t. $R_N\subseteq\bigcup_{j=1}^{J}B_{\varepsilon/2}(x_j)$
~~Given $x\in\Omega$~~ Then $(B_\varepsilon(x_j))_{j=1}^{J}$ is an open ε-cover of Ω since for any $x\in\Omega$, set $x=P_Nx+(I-P_N)x$. Pick j s.t. $P_Nx\in B_{\varepsilon/2}(x_j)$
Then $\|x-x_j\|=\|P_Nx-x_j+(I-P_N)x\|\le\|P_Nx-x_j\|+\|(I-P_N)x\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

<u>Example</u> Consider $H = L^2(\mathbb{T})$ and the set

$$\Omega = \{ f \in H : \|f\|_u \leq C \text{ and } \|f'\|_u \leq C \}$$

Set $\varphi_n(x) = \dfrac{e^{inx}}{\sqrt{2\pi}}$ and

P_N = orthog proj^n onto span$(\varphi_{-N}, \varphi_{-N+1}, \dots, \varphi_{N-1}, \varphi_N)$

We have for $n \neq 0$

$$|\langle \varphi_n, f \rangle| = \left| \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} f(x)\,dx \right| =$$

$$= \left| \left[\frac{e^{-inx}}{-in\sqrt{2\pi}} f(x) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in\sqrt{2\pi}} f'(x)\,dx \right| \leq$$

$$\leq \frac{|f(\pi)| +}{n\sqrt{2\pi}} \frac{|f(-\pi)|}{n\sqrt{2\pi}} + \frac{1}{n\sqrt{2\pi}} \int_{-\pi}^{\pi} |f'(x)|\,dx \leq \frac{C_1}{n}$$

since $\|f\|_u \leq C$ & $\|f'\|_u \leq C$.

NOTE: C_1 does <u>not</u> depend on n or f!

It follows that

$$\|(I - P_N) f\|^2 = \sum_{|n| > N} |\langle \varphi_n, f \rangle|^2 \leq \sum_{|n| > N} \frac{C_1^2}{n^2} \leq \frac{C_2}{N}$$

So for any $\varepsilon > 0$ we can find N

s.t. $\|(I - P_N) f\| \leq \varepsilon$.

It follows that Ω is pre-compact.

Hilbert-Schmidt operators

Review of $H = \mathbb{C}^n$ let A be an $n \times n$ complex matrix.

Recall the def^n of the Frobenius norm of A:

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \left(\text{trace } A^* A \right)^{1/2}$$

Let us compare $\|\cdot\|_F$ to the standard norm $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

$$\|Ax\|^2 = \sum_{i=1}^{n} |r^{(i)} \cdot x|^2 \underset{\text{C-S}}{\leq} \sum_{i=1}^{n} |r^{(i)}|^2 |x|^2 = \|A\|_F^2 \, |x|^2$$

$$A = \begin{bmatrix} - r^{(1)} - \\ - r^{(2)} - \\ \vdots \\ - r^{(n)} - \end{bmatrix}$$

so $\|A\| \leq \|A\|_F$

Homework: (a) Show $\|A\| \leq \|A\|_F \leq \sqrt{n} \, \|A\|$

(b) Find matrices B & C s.t. $\|B\| = \|B\|_F$ (all ones)

$\sqrt{n} \, \|C\| = \|C\|_F$ (identity)

Now let $(\varphi^{(j)})_{j=1}^{n}$ be an ON-basis for H.

We have $\sum_{j=1}^{n} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} |r^{(i)} \cdot \varphi^{(j)}|^2 = \sum_{i=1}^{n} |r^{(i)}|^2 = \|A\|_F$

Alternative def^n: Let $(\varphi^{(j)})_{j=1}^{n}$ be an ON-basis. Set $\|A\|_F = \left(\sum_{j=1}^{n} \|A\varphi^{(j)}\|^2 \right)^{1/2}$

Now suppose A has an ON-basis such that

$A\varphi^{(n)} = \lambda_n \varphi^{(n)}$.

Then $\|A\|_F^2 = \sum_{j=1}^{n} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{n} |\lambda_j|^2$

General Hilbert Space

Lemma Let H be a H.S. and let $A \in \mathcal{B}(H)$.
Let (φ_j) & (ψ_j) be ON-bases for H.
Then $\sum_{j=1}^{\infty} \|A\varphi_j\|^2 = \sum_{j=1}^{\infty} \|A\psi_j\|^2$. (Either both are infinite, or they are both finite and identical.)

Def^n If for some ON-basis (φ_j) it is the case that $\sum \|A\varphi_j\|^2 < \infty$, then we say that H is a Hilbert-Schmidt operator, and define
$$\|A\|_{H\text{-}S} = \left(\sum_{j=1}^{\infty} \|A\varphi_j\|^2 \right)^{1/2}.$$

Note: The def^n does not depend on the choice of basis!

Claim Every H-S operator is compact, but not every compact operator is H.S.

Lemma $\|A\| \leq \|A\|_{H\text{-}S}$.

Lemma If H has an ON-basis (φ_j) s.t. $A\varphi_j = \lambda_j \varphi_j$, then
$$\|A\|_{H\text{-}S} = \left(\sum_{j=1}^{\infty} |\lambda_j|^2 \right)^{1/2}$$

Example $H = \ell^2$ $\quad (\lambda_j)_{j=1}^{\infty}$ is a seq in $\mathbb{C}$.

$Ax = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$

$(e_j)_{j=1}^{\infty}$ is an ON-basis for H s.t. $Ae_j = \lambda_j e_j$

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & - \\ 0 & \lambda_2 & 0 & - \\ 0 & 0 & \lambda_3 & - \\ | & | & | & \end{bmatrix}$$

~~We~~ We have: A is bdd $\Leftrightarrow \sup |\lambda_j| < \infty$

A is compact $\Leftrightarrow |\lambda_j| \to 0$ as $j \to \infty$

A is H-S $\Leftrightarrow \sum_{j=1}^{\infty} |\lambda_j|^2 < \infty$

Example $H = L^2(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$.

$[Au](x) = \int_{\Omega} k(x,y) u(y) \, dA(y)$

$\|A\|_{HS}^2 = \int_{\Omega}\int_{\Omega} |k(x,y)|^2 dA(y)$ so A is H-S $\Leftrightarrow k \in L^2(\Omega^2)$

If A is also S-A, then $\exists\, (\varphi_n)$ & (λ_n) s.t.

$$[Au](x) = \sum \lambda_n \varphi_n(x) (\varphi_n, u) = \sum \lambda_n \varphi_n(x) \int_{\Omega} \varphi_n(y) u(y) \, dA(y) =$$

$$= \int_{\Omega} \left(\sum \lambda_n \varphi_n(x) \overline{\varphi_n(y)} \right) u(y) \, dA(y)$$

so $\quad k(x,y) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \overline{\varphi_n(y)}$

FUNCTIONS OF OPERATORS

Let $f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots + \alpha_d z^d$

be a complex valued polynomial (α_j & $z \in \mathbb{C}$).

It is obvious how to define $f(A)$ for $A \in \mathcal{B}(H)$:

$$f(A) = \alpha_0 + \alpha_1 A + \alpha_2 A^2 + \dots + \alpha_d A^d$$

If A admits a spectral decomp $A = \sum_{n=1}^{\infty} \lambda_n P_n$

where $\underset{\text{if } n \neq m}{P_n P_m = 0}$ & $P_n^2 = P_n$ then $A^k = \sum_{n=1}^{\infty} \lambda_n^k P_n$.

It follows that $f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$.

Now let us generalize slightly to analytic functions.

Let $f(z)$ be analytic on $B_0(R)$, in other words, the sum

$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ converges absolutely for $|z| < R$.

Then if $A \in \mathcal{B}(H)$ is an operator s.t. $\|A\| < R$,

the sum $f_N(A) = \sum_{n=0}^{N} \alpha_n A^n$ converges in norm.

(You can easily prove that it is Cauchy.)

We define $f(A)$ as the limit: $f(A) = \sum_{n=0}^{\infty} \alpha_n A^n$.

<u>Example</u> $f(x) = (1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ for $|z| < R = 1$

$$f(A) = (I - A)^{-1} = \sum_{n=0}^{\infty} A^n \quad \text{for } \|A\| < R = 1$$

Example $f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $|z| < R = \infty$

$$f(A) = \exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \text{ for any } A \in \mathcal{B}(H).$$

If $A = \sum_{n=1}^{\infty} \lambda_n P_n$ and $r(A) = \sup|\lambda_n| < R$, then you can prove that $f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$.

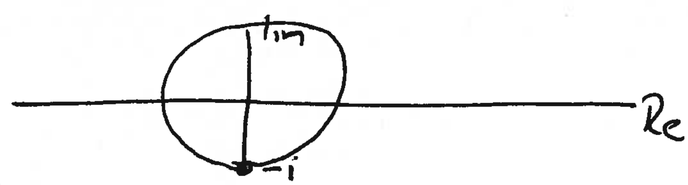
HOWEVER: The sum $f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$ may be norm convergent even when $r(A) > R$.

When it is, we use

$$\boxed{f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n}$$

as the definition of $f(A)$.

Example $f(z) = \frac{z-i}{z+i}$



The radius of analyticity is only 1, so the def^n via power series would work only when $\|A\| < 1$.

But if $A = \sum_{n=1}^{\infty} \lambda_n P_n$ and $|\lambda_n + i| > \delta$ for all n, then $f(A) = \sum_{n=1}^{\infty} \frac{\lambda - i}{\lambda + i} P_n$ is well defined.

~~with $\|f(A)\| \le 1/\delta$ since~~

Note that if $A = \sum_{n=1}^{\infty} \lambda_n P_n$ and $\lambda_n \in \mathbb{R}$ (i.e. A is S-A), then $\left|\frac{\lambda_n - i}{\lambda_n + i}\right| = 1$ so A is unitary if $\sum_{n=1}^{\infty} P_n = I$.

More generally it can be shown that if A is any self-adjoint operator, then $B = (A - iI)(A + iI)^{-1}$ is unitary. (This conversion is known as a Cayley transform.)

General theory (for orientation only).

If A is a normal bounded operator, then A admits a spectral decomposition

$$A = \int_{\sigma(A)} \lambda \, dP(\lambda)$$

where P is a "projection valued measure".

The special case $A = \sum_{n=1}^{\infty} \lambda_n P_n$ occurs when ~~all~~ all of $\sigma(A)$ consists of eigenvalues (with possibly the accumulation point 0 in $\sigma_c(A)$.)

If f is continuous & bdd on $\sigma(A)$ we define

$$f(A) = \int_{\sigma(A)} f(\lambda) \, dP(\lambda).$$

Then $f(\sigma(A)) = \sigma(f(A))$.

Note:
$f(\sigma(A)) \subseteq \{z : |z| = 1\}$ $\Rightarrow$ $f(A)$ is unitary
$f(\sigma(A)) \subset \mathbb{R}$ $\Rightarrow$ $f(A)$ is S-A
$Re(f(\sigma(A))) = 0$ $\Rightarrow$ $f(A)$ is skew-adjoint
$f(A)$ is always normal.