Applied Analysis (APPM 5450): Midterm 3 — Solutions

8.30am – 9.50am, April 18, 2011. Closed books.

Problem 1: In this problem, X denotes a set, and $\mathcal{A}$ denotes a σ-algebra on X.

(a) State the definition of a *measure* μ on $(X, \mathcal{A})$.

(b) Let $(\Omega_j)_{j=1}^{\infty}$ denote a sequence in $\mathcal{A}$ such that $\mu(\Omega_1) < \infty$, and

$$\Omega_1 \supseteq \Omega_2 \supseteq \Omega_3 \supseteq \dots.$$

Set

$$\Omega = \bigcap_{j=1}^{\infty} \Omega_j.$$

Prove that the sequence $(\mu(\Omega_j))_{j=1}^{\infty}$ is convergent, and that $\mu(\Omega) = \lim_{j\to\infty} \mu(\Omega_j)$.

(c) Given an example of a measure space (X, μ) and measurable sets $(\Omega_j)_{j=1}^{\infty}$ such that

$$\Omega_1 \supseteq \Omega_2 \supseteq \Omega_3 \supseteq \dots$$

but $\lim_{j\to\infty} \mu(\Omega_j) \neq \mu\left(\bigcap_{j=1}^{\infty} \Omega_j\right)$.

Solution: We use "$\Cup$" to denote <u>disjoint</u> unions.

(b) Set $A_n = \Omega_n \backslash \Omega_{n+1}$. Then $\Omega_n = A_n \Cup \Omega_{n+1}$ and so

$$\mu(\Omega_n) = \mu(A_n \Cup \Omega_{n+1}) = \mu(A_n) + \mu(\Omega_{n+1}) \geq \mu(\Omega_{n+1}).$$

Since $(\mu(\Omega_n))_{n=1}^{\infty}$ is a decreasing sequence, it must have a limit. To compute the limit, we note that

$$\infty > \mu(\Omega_1) = \mu\left(\Omega \Cup \left(\bigcup_{m=1}^{\infty} A_m\right)\right) = \mu(\Omega) + \sum_{m=1}^{\infty} \mu(A_m).$$

It follows that $\sum_{m=1}^{\infty} \mu(A_m)$ is finite, which implies that $\lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(A_m) = 0$. Finally,

$$\lim_{n\to\infty} \mu(\Omega_n) = \lim_{n\to\infty} \mu\left(\Omega \Cup \left(\bigcup_{m=n}^{\infty} A_m\right)\right) = \lim_{n\to\infty}\left(\mu(\Omega) + \sum_{m=n}^{\infty} \mu(A_m)\right) = \mu(\Omega).$$

(c) Consider $X = \mathbb{R}^2$ with standard Lebesgue measure. Set $\Omega_n = \{x = (x_1, x_2) : |x_2| < 1/n\}$. Then $\mu(\Omega_n) = \infty$ for all n, but $\Omega = \bigcap_{n=1}^{\infty} \Omega_n$ is the x_1-axis, which has measure zero.

Note: The different parts are worth:
(a) 5p
(b) 10p
(c) 10p

Problem 2: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$ be a measurable real-valued function.

(a) State the definition of a Lebesgue integral of f over X.

(b) Consider the special case of $X = \mathbb{R}$ with $\mathcal{A}$ being the power set on $\mathbb{R}$ and

$$\mu(\Omega) = \sum_{j \in \Omega \cap \mathbb{N}} 2^j,$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers. Is μ finite, σ-finite, or neither?

(c) With $(X, \mathcal{A}, \mu)$ as in (b), and with $f(x) = e^{-x}$, evaluate the integral

$$\int_{\mathbb{R}} f \, d\mu.$$

Solution:

(b) We have

$$\mu(\mathbb{R}) = \sum_{j \in \mathbb{R} \cap \mathbb{N}} 2^j = \sum_{j \in \mathbb{N}} 2^j = \sum_{j=1}^{\infty} 2^j = \infty$$

so the measure is not finite. However, if we set $\Omega_j = (j - 1/2, \, j + 1/2]$, then $\{\Omega_j\}_{j \in \mathbb{Z}}$ is a disjoint cover of $\mathbb{R}$, and $\mu(\Omega_j)$ is finite[1] for all j so the measure is σ-finite.

(c)

$$\int_{\mathbb{R}} f \, d\mu = \sum_{j=1}^{\infty} 2^j \, f(j) = \sum_{j=1}^{\infty} 2^j \, e^{-j} = \sum_{j=1}^{\infty} \left(\frac{2}{e}\right)^j = \frac{2/e}{1 - 2/e}.$$

Note: The answer to (c) does not need to be motivated in any detail deeper than that given above. However, to evaluate the integral directly from the definition, first set $g = f \, \chi_{\mathbb{N}}$. Then $f = g$ a.e. so $\int f = \int g$. Now set

$$\varphi_N(x) = \sum_{n=1}^{N} e^{-j} \, \chi_{\{j\}}(x).$$

Then φ_N are simple functions such that $\varphi_N \nearrow g$. Finally

$$\int \varphi_N = \sum_{j=1}^{N} e^{-j} \, 2^j \nearrow \sum_{j=1}^{\infty} e^{-j} \, 2^j = \frac{2/e}{1 - 2/e}.$$

Note: The different parts are worth:
(a) 9p
(b) 8p
(c) 8p

[1] To be precise, $\mu(\Omega_j) = 0$ if $j \leq 0$ and $\mu(\Omega_j) = 2^j$ if $j \in \mathbb{N}$.

Problem 3: No motivation required for parts (a) and (b).

(a) Let $\delta \in \mathcal{S}^*(\mathbb{R})$ denote the Dirac δ-function. What is $\hat{\delta} = \mathcal{F}\delta$?

(b) Let τ_n denote a shift operator on $\mathcal{S}(\mathbb{R})$ defined via $[\tau_n\varphi](x) = \varphi(x-n)$ and generalize to a shift operator on $\mathcal{S}^*(\mathbb{R})$ via $\langle \tau_n T,\, \varphi\rangle = \langle T,\, \tau_{-n}\varphi\rangle$ as usual. Set $T_N = \sum_{n=-N}^{N} \tau_n\,\delta$. What is the Fourier transform $\hat{T}_N$?

(c) Prove that the sequence $(T_N)_{N=1}^{\infty}$ converges in $\mathcal{S}^*(\mathbb{R})$.

(d) Prove that the sequence $(\hat{T}_N)_{N=1}^{\infty}$ converges in $\mathcal{S}^*(\mathbb{R})$.

<u>*5p extra credit:*</u> State the limit point of $(\hat{T}_N)_{N=1}^{\infty}$. No motivation required.

Solution:

(a) $\langle \hat{\delta},\, \varphi\rangle = \langle \delta,\, \hat{\varphi}\rangle = \hat{\varphi}(0) = \displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\,\varphi(x)\,dx$ so $\hat{\delta} = \frac{1}{\sqrt{2\pi}}$.

(b) Recall that $[\mathcal{F}(\tau_n T)](t) = e^{-int}\,\hat{T}(t)$ so

$$\hat{T}_N = \sum_{n=-N}^{N} e^{-int}\frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\frac{e^{i(N+1)t} - e^{-iNt}}{e^{it}-1} = \frac{1}{\sqrt{2\pi}}\frac{e^{i(N+1/2)t} - e^{-i(N+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{1}{\sqrt{2\pi}}\frac{\sin((N+1/2)t)}{\sin(t/2)}.$$

(c) Fix $\varphi \in \mathcal{S}(\mathbb{R})$. We need to prove that the sequence $(T_N(\varphi))_{N=1}^{\infty}$ converges. Observe that

$$|\varphi(x)| \leq \frac{1}{1+x^2}\,\sup_x\bigl((1+x^2)\,|\varphi(x)|\bigr) = \frac{1}{1+x^2}\,||\varphi||_{0,2}.$$

Now

$$T_N(\varphi) = \sum_{n=-N}^{N} [\tau_n\delta](\varphi) = \sum_{n=-N}^{N} \varphi(n).$$

The sum is convergent since $|\varphi(n)| \leq C/(1+n^2)$ and $\sum_{n=-N}^{N} 1/(1+n^2) < \infty$.

(d) Since $\mathcal{F}$ is a continuous map from $\mathcal{S}^*$ to $\mathcal{S}^*$, the fact that (T_N) converges immediately implies that $(\mathcal{F}T_N)$ converges.

Alternative solution to (d):

$$\hat{T}_N(\varphi) = T_N(\hat{\varphi}) = \sum_{n=-N}^{N} \hat{\varphi}(n),$$

and then convergence is proved as in (c) since $\hat{\varphi} \in \mathcal{S}$.

Note: The different parts are worth:
(a) 6p
(b) 6p
(c) 7p
(d) 6p

Comments on the extra credit problem: The distribution T is quite well-known in signal processing and is often called a *Dirac comb*. Its Fourier transform is also a Dirac comb:

$$\hat{T}(t) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \delta(t - 2\pi n). \tag{1}$$

To see this informally (and this is not a rigorous argument!) note that T is a periodic function with period 1, so we can write $T(x) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{i2\pi nx}$. Since $\{e^{i2\pi nx}\}_{n\in\mathbb{Z}}$ is an ON-basis on $L^2(-1/2,\, 1/2)$ we find $\alpha_n = \int_{-1/2}^{1/2} T(x)\, e^{-i2\pi nx}\, dx = 1$, and so

$$T(x) = \sum_{n=-\infty}^{\infty} e^{i2\pi nx}.$$

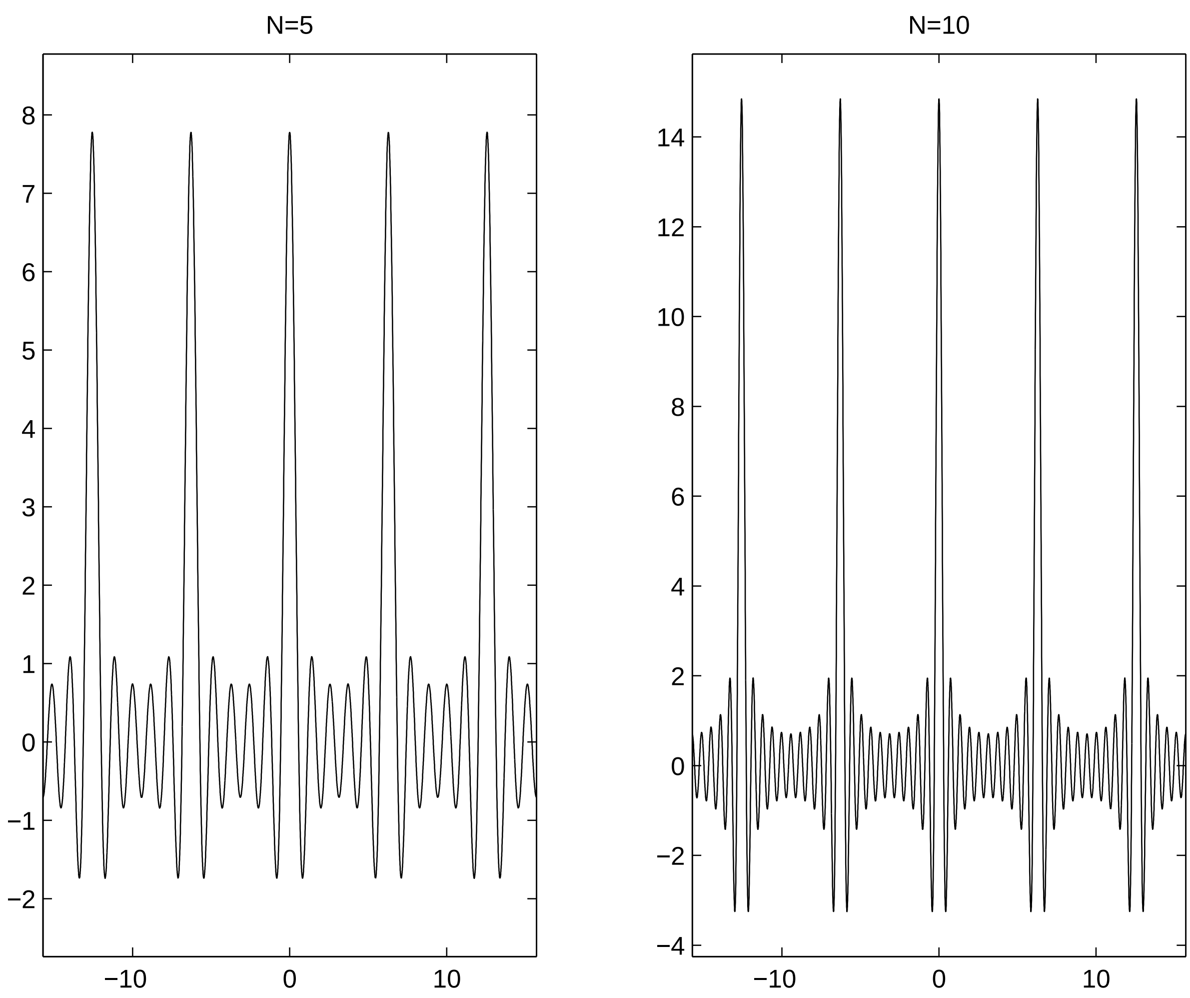
Take the Fourier transform "under the sum" to obtain (1). (Note $[\mathcal{F}e^{i2\pi nx}](t) = \sqrt{2\pi}\, \delta(t - 2\pi n)$.)

From (1) we obtain the important *Poisson summation formula*,

$$\sum_{n=-\infty}^{\infty} \varphi(n) = \langle T,\, \varphi\rangle = \langle \mathcal{F}^*\mathcal{F}T,\, \varphi\rangle = \langle \hat{T},\, \check{\varphi}\rangle = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \check{\varphi}(2\,\pi n) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \hat{\varphi}(2\,\pi n).$$

There is much more on this important and interesting topic in Section 11.11 of the text book.

To illustrate the limit graphically, we plot $\hat{T}_N$ for $N = 5$ and $N = 10$ below:



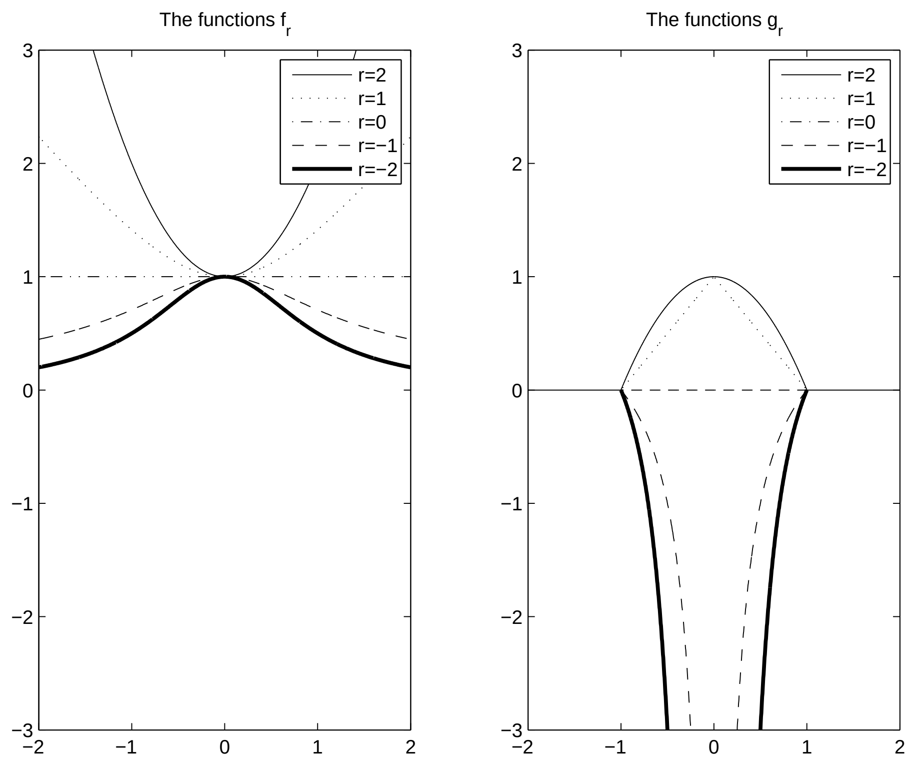
Problem 4: Let r be a real number, and define for $x \in \mathbb{R}\backslash\{0\}$ the functions

$$f_r(x) = (1 + |x|^2)^r, \qquad g_r(x) = 1 - |x|^r.$$

Furthermore, set

$$f_r(0) = 1, \qquad g_r(0) = \begin{cases} 1 & \text{when } r > 0, \\ 0 & \text{when } r \leq 0. \end{cases}$$

The figure below illustrates the definitions:



(a) For which $r \in \mathbb{R}$ is it the case that $f_r \in C_0(\mathbb{R})$?

(b) For which $r \in \mathbb{R}$ is it the case that $g_r \in C_0(\mathbb{R})$?

(c) For which $r \in \mathbb{R}$ is it the case that $\hat{f}_r \in C_0(\mathbb{R})$?

(d) For which $r \in \mathbb{R}$ is it the case that $\hat{g}_r \in C_0(\mathbb{R})$?

(e) For which $r \in \mathbb{R}$ is it the case that $f_r \in \mathcal{S}^*(\mathbb{R})$?

(f) For which $r \in \mathbb{R}$ is it the case that $g_r \in \mathcal{S}^*(\mathbb{R})$?

(g) For which $r \in \mathbb{R}$ is it the case that $\hat{f}_r \in \mathcal{S}^*(\mathbb{R})$?

(h) For which $r \in \mathbb{R}$ is it the case that $\hat{g}_r \in \mathcal{S}^*(\mathbb{R})$?

(i) For which $r \in \mathbb{R}$ and $s \geq 0$ is it the case that $\hat{f}_r \in H^s(\mathbb{R})$?

(j) For which $r \in \mathbb{R}$ and $s \geq 0$ is it the case that $\hat{g}_r \in H^s(\mathbb{R})$?

(Every correct answer will get full credit regardless of whether a motivation is provided.)

5p extra credit: Specify how your answers would change if you consider f_r and g_r as functions on $\mathbb{R}^d$ rather than as functions on $\mathbb{R}$.

Solution to Problem 4:

(a) $\boxed{f_r \in C_0 \text{ if } r < 0.}$
f_r is continuous for all r, and decays iff $r < 0$.

(b) $\boxed{g_r \in C_0 \text{ if } r \geq 0.}$
g_r decays for all r (*super*-fast!), and is continuous iff $r \geq 0$.

(c) $\boxed{\hat{f}_r \in C_0 \text{ if } r < -1/2.}$
By the Riemann-Lebesgue lemma, $\hat{f}_r \in C_0$ if $f_r \in L^1$. Now $\int f_r < \infty$ iff $2r < -1$.

(d) $\boxed{\hat{g}_r \in C_0 \text{ if } r > -1.}$
By the Riemann-Lebesgue lemma, $\hat{g}_r \in C_0$ if $g_r \in L^1$. Now $\int g_r < \infty$ iff $r > -1$.

(e) $\boxed{\text{For all } r.}$
Note that for any r, we have $(1+x^2)^{-r-1}\, f_r = (1+x^2)^{-1} \in L^1$ so f_r is tempered.

(f) $\boxed{\text{For } r > -1.}$
Note that $\int g_r = \infty$ if $r \leq -1$. Conversely, if $r > -1$, then $g_r \in L^1 \subset S^*$.
(Note that decay factors do not help here; *local* integrability is the issue.)

(g) $\boxed{\text{For all } r.}$
Note that $\mathcal{F} : \mathcal{S}^* \to \mathcal{S}^*$ is an isomorphism so $\hat{f}_r \in \mathcal{S}^*$ iff $f_r \in \mathcal{S}^*$. Therefore, the answer must be identical to the answer in (e).

(h) $\boxed{\text{For all } r > -1.}$
Note that $\mathcal{F} : \mathcal{S}^* \to \mathcal{S}^*$ is an isomorphism so $\hat{g}_r \in \mathcal{S}^*$ iff $g_r \in \mathcal{S}^*$. Therefore, the answer must be identical to the answer in (f).

(i) $\boxed{\hat{f}_r \in H^s \text{ iff } 2s + 4r < -1 \text{ (and } s \geq 0).}$
Note that $\hat{f}_r \in H^s$ iff $(1+x^2)^s\, |f_r|^2 \in L^1$ (by definition). Now $(1+x^2)^s\, |f_r(x)|^2 = (1+x^2)^{s+2r}$ which is integrable iff $2s + 4r < -1$.

(j) $\boxed{\hat{g}_r \in H^s \text{ iff } r > -1/2 \text{ (and } s \geq 0).}$
Note that $\hat{g}_r \in H^s$ iff $(1+x^2)^s\, |g_r|^2 \in L^1$ (by definition). Since g_r has compact support, the decay factor is in this case irrelevant, so all that matters is whether $|g_r|^2 \in L^1$. This is true iff $2r > -1$.

Extra credit problems: The answers that change are the ones that depend on integrability. We find:
(c) $r < -d/2$.
(d) $r > -d$.
(f) $r > -d$.
(h) $r > -d$.
(i) $2s + 4r < -d$
(j) $r > -d/2$

Grading guide: Each correct sub-problem is worth 2.5p (with the total rounded up to the nearest integer).