Applied Analysis (APPM 5450): Midterm 3
8.30am – 9.50am, April 18, 2011. Closed books.

Each problem is worth 25 point.

Problem 1: In this problem, X denotes a set, and $\mathcal{A}$ denotes a σ-algebra on X.

(a) State the definition of a *measure* μ on $(X, \mathcal{A})$.

(b) Let $(\Omega_j)_{j=1}^{\infty}$ denote a sequence in $\mathcal{A}$ such that $\mu(\Omega_1) < \infty$, and

$$\Omega_1 \supseteq \Omega_2 \supseteq \Omega_3 \supseteq \dots .$$

Set

$$\Omega = \bigcap_{j=1}^{\infty} \Omega_j.$$

Prove that the sequence $(\mu(\Omega_j))_{j=1}^{\infty}$ is convergent, and that $\mu(\Omega) = \lim_{j\to\infty} \mu(\Omega_j)$.

(c) Given an example of a measure space (X, μ) and measurable sets $(\Omega_j)_{j=1}^{\infty}$ such that

$$\Omega_1 \supseteq \Omega_2 \supseteq \Omega_3 \supseteq \dots$$

but $\lim_{j\to\infty} \mu(\Omega_j) \neq \mu\left(\bigcap_{j=1}^{\infty} \Omega_j\right)$.

Problem 2: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$ be a measurable real-valued function.

(a) State the definition of a Lebesgue integral of f over X.

(b) Consider the special case of $X = \mathbb{R}$ with $\mathcal{A}$ being the power set on $\mathbb{R}$ and

$$\mu(\Omega) = \sum_{j\in\Omega\cap\mathbb{N}} 2^j,$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers. Is μ finite, σ-finite, or neither?

(c) With $(X, \mathcal{A}, \mu)$ as in (b), and with $f(x) = e^{-x}$, evaluate the integral

$$\int_{\mathbb{R}} f \, d\mu.$$

Problem 3: No motivation required for parts (a) and (b).

(a) Let $\delta \in \mathcal{S}^*(\mathbb{R})$ denote the Dirac δ-function. What is $\hat{\delta} = \mathcal{F}\delta$?

(b) Let τ_n denote a shift operator on $\mathcal{S}(\mathbb{R})$ defined via $[\tau_n \varphi](x) = \varphi(x - n)$ and generalize to a shift operator on $\mathcal{S}^*(\mathbb{R})$ via $\langle \tau_n T, \varphi\rangle = \langle T, \tau_{-n} \varphi\rangle$ as usual. Set $T_N = \sum_{n=-N}^{N} \tau_n \delta$. What is the Fourier transform $\hat{T}_N$?

(c) Prove that the sequence $(T_N)_{N=1}^{\infty}$ converges in $\mathcal{S}^*(\mathbb{R})$.

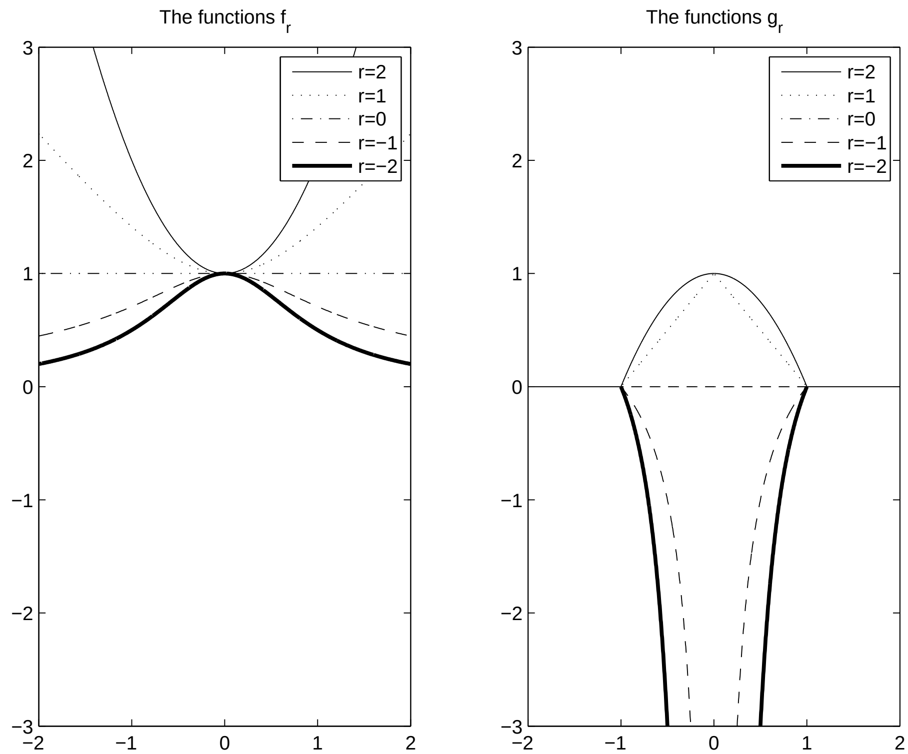
(d) Prove that the sequence $(\hat{T}_N)_{N=1}^{\infty}$ converges in $\mathcal{S}^*(\mathbb{R})$.

5p extra credit: State the limit point of $(\hat{T}_N)_{N=1}^{\infty}$. No motivation required.

Problem 4: Let r be a real number, and define for $x \in \mathbb{R}\backslash\{0\}$ the functions

$$f_r(x) = (1 + |x|^2)^r, \qquad g_r(x) = \begin{cases} 1 & \text{when } x = 0 \text{ and } r > 0, \\ 0 & \text{when } x = 0 \text{ and } r \leq 0, \\ 1 - |x|^r & \text{when } 0 < |x| < 1, \\ 0 & \text{when } 1 < |x|. \end{cases}$$

The figure below illustrates the definitions:



(a) For which $r \in \mathbb{R}$ is it the case that $f_r \in C_0(\mathbb{R})$?

(b) For which $r \in \mathbb{R}$ is it the case that $g_r \in C_0(\mathbb{R})$?

(c) For which $r \in \mathbb{R}$ is it the case that $\hat{f}_r \in C_0(\mathbb{R})$?

(d) For which $r \in \mathbb{R}$ is it the case that $\hat{g}_r \in C_0(\mathbb{R})$?

(e) For which $r \in \mathbb{R}$ is it the case that $f_r \in \mathcal{S}^*(\mathbb{R})$?

(f) For which $r \in \mathbb{R}$ is it the case that $g_r \in \mathcal{S}^*(\mathbb{R})$?

(g) For which $r \in \mathbb{R}$ is it the case that $\hat{f}_r \in \mathcal{S}^*(\mathbb{R})$?

(h) For which $r \in \mathbb{R}$ is it the case that $\hat{g}_r \in \mathcal{S}^*(\mathbb{R})$?

(i) For which $r \in \mathbb{R}$ and $s \geq 0$ is it the case that $\hat{f}_r \in H^s(\mathbb{R})$?

(j) For which $r \in \mathbb{R}$ and $s \geq 0$ is it the case that $\hat{g}_r \in H^s(\mathbb{R})$?

(Every correct answer will get full credit regardless of whether a motivation is provided.)

5p extra credit: Specify how your answers would change if you consider f_r and g_r as functions on $\mathbb{R}^d$ rather than as functions on $\mathbb{R}$.