

## Homework set 7 — APPM5450, Spring 2011 — partial solutions

**Problem 9.21:** Suppose  $A \in \mathcal{B}(H)$  is such that

$$\operatorname{Re}(x, Ax) \leq 2\alpha \|x\|^2.$$

Prove that the solution  $x = x(t)$  of  $x'(t) = Ax(t)$  satisfies

$$\|x(t)\| \leq e^{\alpha t} \|x(0)\|.$$

*Note:* The book may have a typo — the bound seems off by a factor of two. Consider for instance  $Ax = 2\alpha x$ , then  $x(t) = e^{2\alpha t}x(0)$ .

*Solution:* Set  $f(t) = \|x(t)\|^2$ . Then

$$f'(t) = \frac{d}{dt}(x, x) = (x', x) + (x, x') = (Ax, x) + (x, Ax) = 2\operatorname{Re}(x, Ax) \leq 4\alpha \|x(t)\|^2 = 4\alpha f(t).$$

By the Grönwall inequality, we find

$$\|x(t)\|^2 = f(t) \leq f(0) \exp\left(\int_0^t 4\alpha ds\right) = f(0) e^{4\alpha t} = \|x(0)\|^2 e^{4\alpha t}.$$

Extract the square root to obtain the desired bound.

**Problem 9.22:** Let  $A$  be compact and non-negative. Prove that there exists a unique compact non-negative operator  $B$  such that  $B^2 = A$ .

*Solution:* Since  $A$  is self-adjoint and compact, there is an ON-basis  $(\varphi_n)_{n=1}^\infty$  of eigen-vectors of  $A$ .  $A\varphi_n = \lambda_n \varphi_n$ . We know  $|\lambda_n| \rightarrow 0$  since  $A$  is compact, and  $\lambda_n \geq 0$  since  $A$  is non-negative.

Existence: Set  $B = \sum_{n=1}^\infty \sqrt{\lambda_n} P_n$  where  $P_n x = (\varphi_n, x) \varphi_n$ . It is easily shown that  $B^2 = A$  and that  $B$  is compact and non-negative.

Observe that from the construction of  $B$ , it follows that if  $\psi$  is a vector such that  $A\psi = \lambda\psi$ , then  $B\psi = \sqrt{\lambda}\psi$ .

Uniqueness: Suppose that  $C$  is a non-negative compact operator such that  $C^2 = A$ . We need to show that  $C = B$ , where  $B$  is the operator constructed above. Since  $C$  is compact and self-adjoint, there is an ON-basis  $(\psi_n)_{n=1}^\infty$  such that  $C\psi_n = \mu_n \psi_n$ . Now observe that

$$A\psi_n = C^2\psi_n = C(\mu_n \psi_n) = \mu_n^2 \psi_n$$

so  $\psi_n$  is an eigenvector of  $A$  with eigenvalue  $\mu_n^2$ . It follows that  $B\psi_n = \sqrt{\mu_n^2} \psi_n = \mu_n \psi_n = C\psi_n$ . (We know that  $\sqrt{\mu_n^2} = \mu_n$  since  $C$  must be non-negative, which implies that  $\mu_n \geq 0$ .)

**Problem 1:** Consider the Hilbert space  $H = \mathbb{C}^n$ . Let  $A \in \mathcal{B}(H)$ , let  $(e^{(j)})_{j=1}^n$  be the canonical basis, and let  $A$  have the representation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

in the canonical basis. We define the *Hilbert-Schmidt norm* of  $A$  as

$$\|A\|_{\text{HS}} = \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

(a) Let  $(\varphi^{(j)})_{j=1}^n$  be any ON-basis for  $H$ . Show that  $\|A\|_{\text{HS}}^2 = \sum_{j=1}^n \|A\varphi^{(j)}\|^2$ .

(b) Show that  $\|A\| \leq \|A\|_{\text{HS}} \leq \sqrt{n}\|A\|$  for any  $A \in \mathcal{B}(H)$ .

(c) Find  $G, H \in \mathcal{B}(H)$  such that  $\|G\|_{\text{HS}} = \|G\|$  and  $\|H\|_{\text{HS}} = \sqrt{n}\|H\|$ .

*Solution:*

(a) Let  $r^{(i)}$  denote the  $i$ 'th row of  $A$ . Then

$$\sum_{j=1}^n \|A\varphi^{(j)}\|^2 = \sum_{j=1}^n \sum_{i=1}^n \|(r^{(i)}, \phi^{(j)})\|^2 = \{\text{Parseval}\} = \sum_{i=1}^n \|r^{(i)}\|^2 = \|A\|_{\text{HS}}^2.$$

(b) For any  $x$  a simply application of Cauchy-Schwartz yields

$$\|Ax\|^2 = \sum_{i=1}^n \|(r^{(i)}, x)\|^2 \leq \sum_{i=1}^n \|r^{(i)}\|^2 \|x\|^2 = \|A\|_{\text{HS}}^2 \|x\|^2.$$

It follows that  $\|A\| \leq \|A\|_{\text{HS}}$ . Next, let  $i$  be such that  $\|r^{(i)}\| = \max_j \|r^{(j)}\|$ . Then

$$\|A\|_{\text{HS}}^2 = \sum_{j=1}^n \|r^{(j)}\|^2 \leq n \|r^{(i)}\|^2 = n \|A^* e_i\|^2 \leq n \|A^*\| = n \|A\|,$$

where  $e_i$  denotes the  $i$ 'th canonical basis vector.

(c) For instance, let  $G$  be the matrix consisting of all ones, and let  $H$  be the identity matrix.

**Problem 2:** Let  $H$  be a separable Hilbert space, and let  $A \in \mathcal{B}(H)$ . Suppose that  $H$  has an ON-basis  $(\varphi^{(j)})_{j=1}^{\infty}$  such that

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 < \infty.$$

Prove that if  $(\psi^{(j)})_{j=1}^{\infty}$  is any other ON-basis, then

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{\infty} \|A\psi^{(j)}\|^2.$$

*Solution:* Set

$$\alpha_{ji} = (A\varphi^{(j)}, \psi^{(i)}) = (\varphi^{(j)}, A^*\psi^{(i)})$$

and

$$\beta_{ik} = (A^*\psi^{(i)}, \psi^{(k)}) = (\psi^{(i)}, A\psi^{(k)}).$$

The proof consists of four applications of Parseval:

$$\sum_{j=1}^{\infty} \|A\varphi^{(j)}\|^2 = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\alpha_{ji}|^2 = \sum_{i=1}^{\infty} \|A^*\psi^{(i)}\|^2 = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |\beta_{ik}|^2 = \sum_{k=1}^{\infty} \|A\psi^{(k)}\|^2.$$

Note that the interchanges of summation order are permissible as all terms are non-negative.

**Problem 3:** Consider the linear space  $L = \mathbb{R}^2$ . Define for  $x = (x_1, x_2) \in L$  the seminorms

$$p_1(x) = |x_1|, \quad p_2(x) = |x_2|.$$

Construct for  $x \in L$ ,  $j \in \{1, 2\}$ , and  $\varepsilon \in (0, \infty)$ , the sets

$$\mathcal{B}_{x,j,\varepsilon} = \{y \in L : p_j(x - y) < \varepsilon\}.$$

Describe these sets geometrically. What is the topology generated by the collection of semi-norms  $\{p_1\}$ ? Is it Hausdorff? What is the topology generated by the collection of semi-norms  $\{p_1, p_2\}$ ? Is it Hausdorff?

*Solution:*

For  $x = (x_1, x_2)$ , the set  $\mathcal{B}_{x,1,\varepsilon}$  is a vertical strip of width  $2\varepsilon$  centered around  $x_1$ . The set  $\mathcal{B}_{x,2,\varepsilon}$  is a horizontal strip of width  $2\varepsilon$  centered around  $x_2$ .

The topology  $\mathcal{T}_1$  generated by  $\{p_1\}$  is the topology on the real line. In other words,  $\Omega \in \mathcal{T}_1$  iff  $\Omega = \Omega_1 \times \mathbb{R}$  where  $\Omega_1$  is an open set on the line. This topology is not Hausdorff. For a counter-example, set  $x = (0, 0)$  and  $y = (0, 1)$ . Then if  $\Omega \in \mathcal{T}_1$  we have

$$x \in \Omega \quad \Leftrightarrow \quad y \in \Omega.$$

As far as  $\mathcal{T}_1$  is concerned, the points  $x$  and  $y$  are not distinct.

The topology generated by  $\{p_1, p_2\}$  has as its base  $\mathcal{B}$  intersections of open sets in  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . This means that  $\mathcal{B}$  consists of all open rectangles in the plane. These generate the standard topology on  $\mathbb{R}^2$ , which is Hausdorff.