Homework set 7 — APPM5450, Spring 2011

From the text-book: 9.19, 9.20, 9.22. Optional: 9.21.

Problem 1: Consider the Hilbert space $H = \mathbb{C}^n$. Let $A \in \mathcal{B}(H)$, let $(e^{(j)})_{j=1}^n$ be the canonical basis, and let A have the representation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

in the canonical basis. We define the *Hilbert-Schmidt norm* of A as

$$||A||_{\text{HS}} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

(a) Let $(\varphi^{(j)})_{j=1}^n$ be any ON-basis for H. Show that $||A||_{\text{HS}}^2 = \sum_{j=1}^{n} ||A\varphi^{(j)}||^2$.

(b) Show that $||A|| \leq ||A||_{\text{HS}} \leq \sqrt{n}||A||$ for any $A \in \mathcal{B}(H)$.

(c) Find $G, H \in \mathcal{B}(H)$ such that $||G||_{\text{HS}} = ||G||$ and $||H||_{\text{HS}} = \sqrt{n}||H||$.

Problem 2: Let H be a separable Hilbert space, and let $A \in \mathcal{B}(H)$. Suppose that H has an ON-basis $(\varphi^{(j)})_{j=1}^\infty$ such that

$$\sum_{j=1}^{\infty} ||A\varphi^{(j)}||^2 < \infty.$$

Prove that if $(\psi^{(j)})_{j=1}^\infty$ is any other ON-basis, then

$$\sum_{j=1}^{\infty} ||A\varphi^{(j)}||^2 = \sum_{j=1}^{\infty} ||A\psi^{(j)}||^2.$$

Problem 3: Consider the linear space $L = \mathbb{R}^2$. Define for $x = (x_1,\, x_2) \in L$ the seminorms

$$p_1(x) = |x_1|, \qquad p_2(x) = |x_2|.$$

Construct for $x \in L$, $j \in \{1,\, 2\}$, and $\varepsilon \in (0, \infty)$, the sets

$$\mathcal{B}_{x,j,\varepsilon} = \{y \in L : \, p_j j(x - y) < \varepsilon\}.$$

Describe these sets geometrically. What is the topology generated by the collection of semi-norms $\{p_1\}$? Is it Hausdorff? What is the topology generated by the collection of semi-norms $\{p_1,\, p_2\}$? Is it Hausdorff?