Homework 6

1) Let H_1 and H_2 be Hilbert spaces, let $U : H_1 \to H_2$ be unitary, and let $A \in B(H_1)$. Define $\widetilde{A} \in B(H_2)$ by $\widetilde{A} = UAU^{-1}$. Prove the following.

a) $\sigma_p(A) = \sigma_p(\widetilde{A})$

Assume $\lambda \in \sigma_p(A)$, then for some $x \in H_1$ we have $Ax = \lambda x$. There exists $x' \in H_2$ s.t. $Ux = x'$ (and also $U^{-1}x' = Ux$).

Then $\lambda x' = \lambda \underbrace{UU^{-1}}_{I} x' = U\lambda \underbrace{U^{-1}x'}_{x} = UA\underbrace{U^{-1}x'}_{x} = \widetilde{A}x'$.

Here the first equality uses $UU^{-1} = I$.

The second equality simply moves the constant.

The third equality uses $Ax = \lambda x$ (which we assumed).

The final equality rewrites $\widetilde{A} = UAU^{-1}$.

Note that we can easily prove the other direction with an almost identical argument.

b) $\sigma_c(A) = \sigma_c(\widetilde{A})$

Assume $\lambda \in \sigma_c(A)$.

Then $(A - \lambda I)$ is one-to-one. This implies that $U(A - \lambda I)U^{-1} = (\widetilde{A} - \lambda I)$ is also one-to-one.

Also, $ran(A - \lambda I)$ is dense. We want to prove $ran(\widetilde{A} - \lambda I) = ran(U(A - \lambda I)U^{-1})$ is dense.

Pick $x \in H_2$ and set $x' = U^{-1}x \in H_1$.

Since $ran(A - \lambda I)$ is dense there exist $y_n' \in H_1$ s.t. $(A - \lambda I)y_n' \to x'$ in H_1. Set $y_n = Uy_n'$.

Then $U(A - \lambda I)\underbrace{y_n'}_{=U^{-1}y_n} \to \underbrace{Ux'}_{=x}$ in H_2. Then $U(A - \lambda I)U^{-1}y_n \to x$ in H_2.

c) $\sigma_r(A) = \sigma_r(\widetilde{A})$

Assume $\lambda \in \sigma_c(A)$.

Then $(A - \lambda I)$ is one-to-one. This implies that $U(A - \lambda I)U^{-1} = (\widetilde{A} - \lambda I)$ is also one-to-one.

There exists an $x \in ran(A - \lambda I)^{\perp}$ (i.e there exists an x s.t. (for all y) $0 = \langle (A - \lambda I)y, x \rangle$).

We want to prove that there exists an $x' \in ran(\widetilde{A} - \lambda I)^{\perp} = ran(U(A - \lambda I)U^{-1})^{\perp}$.

Set $x' = Ux$ and $y' = Uy$.

Then $0 = \langle (A - \lambda I)y, x \rangle \Rightarrow 0 = \langle U(A - \lambda I)y, Ux \rangle = \langle U(A - \lambda I)U^{-1}y', x' \rangle$.

2) Let A be a self-adjoint compact operator. For $\lambda \in \rho(A)$ set $R_\lambda = (A - \lambda I)^{-1}$ as usual. Construct the spectral decomposition of R_λ.

Use it to prove that : $\|R_\lambda\| = \dfrac{1}{dist(\lambda, \sigma(A))} = \dfrac{1}{\inf_{\mu \in \sigma(A)} |\lambda - \mu|}$.

Since A is a compact, self-adjoint we can write $A = \sum_{n=1}^{\infty} \lambda_n P_n$ where $|\lambda_n| \to 0$ and P_n are mutually orthogonal projections.

Then $(A - \lambda I)^{-1} = \sum_{n=1}^{\infty} \dfrac{1}{\lambda_n - \lambda} P_n$.

Then $\|(A - \lambda I)^{-1}\| = \sup_n \left| \dfrac{1}{\lambda_n - \lambda} \right| = \dfrac{1}{\inf_n |\lambda_n - \lambda|} = \dfrac{1}{dist(\lambda, \sigma(A))}$.

3) Consider the Hilbert space $H = L^2(I)$ where $I = [-\pi, \pi]$.
Define $\Omega_t = \{u \in H : u(x) = 0 \, \forall x \geq t\}$ (note that this is a closed linear subspace of H).
Define $P(t)$ as the orthogonal projection onto Ω_t.
Consider the operator $A \in B(H)$ defined by $[Au](x) = xu(x)$.

a) Prove that Ω_t is an invariant subspace of A for every $t \in R$.

Note that M is an invariant subspace of A if $Au \in M \ \forall u \in M$.
Pick a $u \in \Omega_t$. Then $[Au](x) = xu(x)$.
We need to show that for any $u(x) \in \Omega_t$ we have $xu(x) \in \Omega_t$.
For any $x \geq t$ we know that $u(x) = 0 \Rightarrow xu(x) = 0$ so $xu(x) \in \Omega_t$.

b) Prove that if $a < b \leq c < d$ then $ran(P(b) - P(a)) \perp ran(P(d) - P(c))$.
Conclude that for any numbers $-\pi = t_0 < t_1 < t_2 < \cdots < t_n = \pi$ it is the case that
$H = ran[P(t_1) - P(t_0)] \oplus ran[P(t_2) - P(t_1)] \oplus \cdots \oplus ran[P(t_n) - P(t_{n-1})]$ where each term is an invariant subspace of A.

Note that $ranP(t) = \{u \in H : u(x) = 0 \ \forall x \geq t\}$.
Then $ran(P(b) - P(a)) = \{u \in H : u(x) = 0 \ \forall x \leq a, x \geq b\}$ (i.e. $supp(P(b) - P(a)) \subseteq [a,b]$).
Similarly $ran(P(d) - P(c)) = \{u \in H : u(x) = 0 \ \forall x \leq c, x \geq d\}$ (i.e. $supp(P(d) - P(c)) \subseteq [c,d]$).
Since $a < b \leq c < d$ these supports are disjoint and $ran(P(b) - P(a)) \perp ran(P(d) - P(c))$.

This easily generalizes to the case where we have n separate supports (as in the later part of this problem). Each term is an invariant subspace because they are projections onto a region (the proof is just as simple as in part (a), just replace $[-\pi, t]$ with $[t_j, t_{j-1}]$).

c) For a positive integer n set $h = 2\pi/n$ and $\lambda_j = -\pi + hj$.

Define the operator $A_n = \sum_{j=1}^{n} \lambda_j \left(P(\lambda_j) - P(\lambda_{j-1})\right)$. Prove that $\|A - A_n\| < 2\pi/n$.

Conclude that $A_n \to A$ in norm.

$$\|A - A_n\|^2 = \sup_{u \in H} \left| \int_{-\pi}^{\pi} \left((A - A_n)u(x)\right)^2 dx \right| = \sup_{u \in H} \left| \int_{-\pi}^{\pi} \left(xu(x) - \sum_{j=1}^{n} \lambda_j \left(P(\lambda_j) - P(\lambda_{j-1})\right)u(x) \right) dx \right| =$$

$$= \sup_{u \in H} \left| \sum_{j=1}^{n} \int_{\lambda_{j-1}}^{\lambda_j} \left(xu(x) - \lambda_j \left(P(\lambda_j) - P(\lambda_{j-1})\right)u(x)\right)^2 \right| = \sup_{u \in H} \left| \sum_{j=1}^{n} \int_{\lambda_{j-1}}^{\lambda_j} \left(xu(x) - \lambda_j \tilde{u}(x)\right)^2 \right| = \sup_{u \in H} \left| \sum_{j=1}^{n} \int_{\lambda_{j-1}}^{\lambda_j} \left((x - \lambda_j)u(x)\right)^2 \right| \leq$$

$$\leq \sup_{u \in H} \left| \sum_{j=1}^{n} \left(\underbrace{\sup_{x \in [\lambda_j, \lambda_{j-1}]} |x - \lambda_j|^2}_{=h^2} \int_{\lambda_{j-1}}^{\lambda_j} (u(x))^2 \right) \right| = h^2 \|u\|^2$$

$$\Rightarrow \|A - A_n\|^2 \leq h^2 = (2\pi/n)^2 = 4\pi^2/n^2 \Rightarrow \|A - A_n\| \leq 2\pi/n$$

The first equality is the definition of the norm.

The second equality substitutes in the definitions for A and A_n.

The equality across the first line break exchanges the summation and integral (valid because the sum has a finite number of terms).

The second equality on the middle line substitutes $\left(P(\lambda_j) - P(\lambda_{j-1})\right)u(x) = \tilde{u}(x) = u(x)\mathrm{X}_{[\lambda_{j-1}, \lambda_j]}$.

The next equality uses the fact that $\tilde{u}(x) = \begin{cases} u(x) & \lambda_{j-1} \leq x \leq \lambda_j \\ 0 & else \end{cases}$.

The inequality (across the line break) factors out the coefficients in each segment.

The final equality uses $\sup_{x \in [\lambda_j, \lambda_{j-1}]} |x - \lambda_j| \leq h$.

The last line combines everything to complete the proof.