

## Homework set 6 — APPM5450, Spring 2011

If you didn't complete all of the problems 9.1 – 9.11 last week, then continue working on that.

**Problem 1:** Let  $H_1$  and  $H_2$  be Hilbert spaces, let  $U : H_1 \rightarrow H_2$  be unitary, and let  $A \in \mathcal{B}(H_1)$ . Define  $\tilde{A} \in \mathcal{B}(H_2)$  by  $\tilde{A} = U A U^{-1}$ . Prove that

- $\sigma_p(A) = \sigma_p(\tilde{A})$
- $\sigma_c(A) = \sigma_c(\tilde{A})$
- $\sigma_r(A) = \sigma_r(\tilde{A})$

**Problem 2:** Let  $A$  be a self-adjoint compact operator. For  $\lambda \in \rho(A)$ , set  $R_\lambda = (A - \lambda I)^{-1}$  as usual. Construct the spectral decomposition of  $R_\lambda$ . Use it to prove that

$$\|R_\lambda\| = \frac{1}{\text{dist}(\lambda, \sigma(A))} = \frac{1}{\inf_{\mu \in \sigma(A)} |\lambda - \mu|}.$$

**Problem 3:** Consider the Hilbert space  $H = L^2(I)$ , where  $I = [-\pi, \pi]$ . Define

$$\Omega_t = \{u \in H : u(x) = 0 \forall x \geq t\}.$$

Note that  $\Omega_t$  is a closed linear subspace of  $H$ . Define  $P(t)$  as the orthogonal projection onto  $\Omega_t$ . Consider the self-adjoint operator  $A \in \mathcal{B}(H)$  defined by

$$[Au](x) = x u(x).$$

(a) Prove that  $\Omega_t$  is an invariant subspace of  $A$  for every  $t \in \mathbb{R}$ .

(b) Prove that if  $a < b \leq c < d$ , then  $\text{ran}(P(b) - P(a)) \perp \text{ran}(P(d) - P(c))$ . Conclude that for any numbers  $-\pi = t_0 < t_1 < t_2 < \dots < t_n = \pi$ , it is the case that

$$H = \text{ran}[P(t_1) - P(t_0)] \oplus \text{ran}[P(t_2) - P(t_1)] \oplus \dots \oplus \text{ran}[P(t_n) - P(t_{n-1})],$$

where each term is an invariant subspace of  $A$ .

(c) For a positive integer  $n$ , set  $h = 2\pi/n$ , and  $\lambda_j = -\pi + h j$ . Define the operator

$$(1) \quad A_n = \sum_{j=1}^n \lambda_j (P(\lambda_j) - P(\lambda_{j-1})).$$

Prove that  $\|A - A_n\| \leq 2\pi/n$ . Conclude that  $A_n \rightarrow A$  in norm.

**Note:** There is a spectral theorem for all self-adjoint operators on Hilbert spaces (all normal ones, even). For operators with continuum spectra such as  $A$ , the spectral decomposition of  $A$  involves so called “projection valued measures”. The sum (1) is a Riemann-Stieltjes sum of the integral

$$A = \int_{-\pi}^{\pi} \lambda dP(\lambda).$$