9.10 (slightly reworded): Consider the Hilbert space $H = l^2(\mathbb{N})$ and the operators L and R defined by

$$R\,(x_1,\, x_2,\, x_3,\, \dots) = (0,\, x_1,\, x_2,\, \dots),$$
$$L\,(x_1,\, x_2,\, x_3,\, \dots) = (x_2,\, x_3,\, x_4,\, \dots).$$

Determine and classify the spectra of L and R.

Solution: First we determine the eigenvalues. Let λ be a complex number and consider the equation $R\,x = \lambda\,x$. It is easily seen that the only solution is $x = 0$ (ones needs to distinguish between the cases $\lambda = 0$ and $\lambda \neq 0$), and so $\sigma_{\mathrm{p}}(R) = \emptyset$. Next consider the equation $L\,x = \lambda\,x$. The only solution is $x = x_1\,(1,\, \lambda,\, \lambda^2, \dots)$, so we can construct eigenvectors in H if and only if $|\lambda| < 1$. Thus $\sigma_{\mathrm{p}}(L) = \{\lambda :\ |\lambda| < 1\}$.

<u>Consider λ such that $|\lambda| > 1$:</u> Since $||L|| = ||R|| = 1$, we immediately find that $\lambda \notin \sigma(L)$ and $\lambda \notin \sigma(R)$.

<u>Consider λ such that $|\lambda| < 1$:</u> We already determined that $\lambda \in \sigma_{\mathrm{p}}(L)$. Since $R = L^*$, this fact gives us information about the spectrum of R as well since

$$\overline{\mathrm{ran}(R - \lambda I)} = \big(\ker(R^* - \bar{\lambda} I)\big)^{\perp} = \big(\ker(L - \bar{\lambda} I)\big)^{\perp} \neq \{0\}.$$

It follows that $\lambda \in \sigma(R)$, but that $\lambda \notin \sigma_{\mathrm{c}}(R)$. Moreover, we determined previously that $\sigma_{\mathrm{p}}(R) = \emptyset$, so we must have that $\lambda \in \sigma_{\mathrm{r}}(R)$.

<u>Consider λ such that $|\lambda| = 1$:</u> First note that since the spectrum of any operator is closed, and the open unit disc belongs to the spectra of both L and R, we know that λ belongs to both $\sigma(L)$ and $\sigma(R)$. We also know that λ is not an eigenvalue of either L or R. Suppose that λ is in the residual spectrum of L. It would then follow that $\bar{\lambda} \in \sigma_{\mathrm{p}}(L^*) = \sigma_{\mathrm{p}}(R)$, but we know that this is not the case. Thus $\lambda \in \sigma_{\mathrm{c}}(L)$. The proof that $\lambda \in \sigma_{\mathrm{c}}(R)$ is analogous.