

Homework set 4 — APPM5450, Spring 2011

From the textbook: 8.15.

Problem 1: Consider the Hilbert space $H = l^2(\mathbb{N})$; let e_n denote the canonical basis vectors. Which of the following sequences converge weakly? Which have convergent subsequences?

- (a) $x_n = n e_n$.
- (b) $y_n = n^{-1/2} \sum_{j=1}^n e_j$.
- (c) $x_n = e_n + e_m$ where $m = 1 + \text{mod}(n, 2)$.

Problem 2: Consider the Hilbert space $H = L^2([-\pi, \pi])$, and the sequence of functions $\varphi_n(x) = x^2 \sin(nx)$. Does $(\varphi_n)_{n=1}^\infty$ converge strongly in H ? Does $(\varphi_n)_{n=1}^\infty$ converge weakly in H ? If you answer yes to either question, specify the limit.

Problem 3: Let A denote a self-adjoint operator on a Hilbert space H . Let u denote an element of H and set $u_n = e^{inA} u$. Prove that $(u_n)_{n=1}^\infty$ has a weakly convergent subsequence.

Problem 4: Let H_1 and H_2 be two Hilbert spaces. Let $U : H_1 \rightarrow H_2$ be a unitary operator, and let $A_1 \in \mathcal{B}(H_1)$ be a self-adjoint operator. Define the operator $A_2 \in \mathcal{B}(H_2)$ by $A_2 = U A_1 U^{-1}$. Prove that A_2 is self-adjoint.

Problem 5 (optional): Consider the Hilbert space $H = L^2([-\pi, \pi])$, and let \mathcal{P} denote the set of trigonometric polynomials (which is dense in H). For $u \in \mathcal{P}$, let A denote the operator $Au = 100u + 18u'' + u''''$. Prove that

$$\sup_{u \in \mathcal{P}, \|u\|=1} \langle Au, u \rangle = \infty.$$

Conclude that A cannot be extended to a bounded linear operator on H . Prove that for $u, v \in \mathcal{P}$, $\langle Au, v \rangle = \langle u, Av \rangle$. Determine

$$\inf_{u \in \mathcal{P}, \|u\|=1} \langle Au, u \rangle.$$

Prove that

$$\langle u, v \rangle_A = \langle Au, v \rangle$$

is a bilinear form on \mathcal{P} . Prove that on \mathcal{P} , the norm $\|\cdot\|_A$ induced by $\langle \cdot, \cdot \rangle_A$ is equivalent to the norm

$$\|u\|_{H^2(\mathbb{T})} = \sqrt{\|u\|_{L^2(\mathbb{T})}^2 + \|u''\|_{L^2(\mathbb{T})}^2}.$$

Conclude that the closure of \mathcal{P} under the norm $\|\cdot\|_A$ is the space $H^2(\mathbb{T})$ (as defined in Section 7.2).