**Homework set 14 — APPM5450, Spring 2011 — Solutions**

**Problem 12.8:** We want to prove that

$$||f - f_n||_p^p = \int |f - f_n|^p \to \infty.$$

We know that $|f - f_n|^p \to 0$ pointwise, so if we can only justify moving the limit inside the integral, we'll be done.

First note that

$$|f(x)| = \lim_{n\to\infty} |f_n(x)| \le |g(x)|.$$

Then we can dominate the integrand as follows:

$$|f - f_n|^p \le (|f| + |f_n|)^p \le (|g| + |g|)^p \le 2^p |g|^p.$$

Since $\int |g|^p < \infty$, we find that the Lebesque dominated convergence theorem applies, and so

$$\lim_{n\to\infty} ||f - f_n||_p^p = \lim_{n\to\infty} \int |f - f_n|^p = \{\text{LDCT}\} = \int \big(\lim_{n\to\infty} |f - f_n|^p\big) = \int 0 = 0.$$

**Problem 12.16:** Fix $f \in L^p$ and $\varepsilon > 0$. We want to prove that there exists a $\delta > 0$ such that for $|h| < \delta$, we have $||f - \tau_h f||_p < \varepsilon$.

First pick $\varphi \in C_{\rm c}$ such that $||f - \varphi||_p < \varepsilon/3$. Then

$$\begin{aligned}||f - \tau_h f||_p &\le ||f - \varphi||_p + ||\varphi - \tau_h\varphi||_p + ||\tau_h\varphi - \tau_h f||_p \\ &= ||f - \varphi||_p + ||\varphi - \tau_h\varphi||_p + ||\varphi - f||_p < \varepsilon/3 + ||\varphi - \tau_h\varphi||_p + \varepsilon/3.\end{aligned}$$

Set $R = \sup\{|x| : \varphi(x) \ne 0\}$. Since $\varphi$ is uniformly continuous, there exists a $\delta$ such that if $|x - y| < \delta$, then $|\varphi(x) - \varphi(y)| < \varepsilon/(3\mu(B_{R+1}(0))^{1/p})$. Then, if $h < \min(\delta, 1)$,

$$||\varphi - \tau_h\varphi||_p^p = \int_{B_{R+1}(0)} |\varphi(x) - \varphi(x - h)|^p\,dx < \int_{B_{R+1}(0)} \frac{\varepsilon^p}{3^p\mu(B_{R+1}(0))}\,dx < \frac{\varepsilon^p}{3^p}.$$

**Problem 12.17:** For $n = 1, 2, 3, \ldots$, set $I_n = (2^{-n}, 2^{-n+1})$, and $f_n = 2^{n/p}\,\chi_{I_n}$. Then $||f_n||_p = 1$ for all $n$. Suppose $m \ne n$, then

$$||f_n - f_m||_\infty = 1,$$

and for $p \in [1, \infty)$ we have

$$||f_n - f_m||_p = \left(\int_0^1 (2^n\chi_{I_n} + 2^m\chi_{I_m})\right)^{1/p} = 2^{1/p}.$$

No subsequence of $(f_n)_{n=1}^\infty$ can be Cauchy, and therefore no subsequence can converge.

**Problem 12.18:** For $n = 1, 2, 3, \ldots$, set $I_n = (2^{-n}, 2^{-n+1})$, and $f_n = 2^n\,\chi_{I_n}$. Let $(f_{n_j})_{j=1}^\infty$ be a subsequence of $(f_n)_{n=1}^\infty$. Define $g \in L^\infty$ by

$$g = \sum_{j=1}^\infty (-1)^j \chi_{I_{n_j}},$$

and define $\varphi \in (L^1)^*$ via $\varphi(f) = \int fg$. Then $\varphi(f_{n_j}) = (-1)^j$ (verify!) and so $(f_{n_j})$ cannot converge weakly. Since $L^1$ is not reflexive, this does not contradict that Banach-Alaoglu theorem.

**Problem 12.13:** Set $I = [0, 1]$ and let $\Omega$ be a dense set in $L^\infty(I)$. For $r \in I$, set $f_r = \chi_{[0,\,r]}$, and pick $x_r \in \Omega \cap B_{1/3}(f_r)$. Since $||f_r - f_s|| = 1$ if $s \neq r$, we find that $||x_r - x_s|| \geq ||f_r - f_s|| - ||f_r - x_r|| - ||f_s - x_s|| \geq 1/3$, so all the $x_r$'s are distinct. Therefore, $\Omega$ must be uncountable, and $L^\infty$ cannot be seperable.

To prove that $C(I)$ cannot be dense in $L^\infty(I)$, simply note that if $f = \chi_{[0,1/2]}$, and $\varphi \in C(I)$, then

$$||f - \varphi||_\infty \geq \max(|\varphi(1/2)|,\, |1 - \varphi(1/2)|) \geq 1/2$$

(verify this!).

An alternative argument for why $C(I)$ cannot be dense in $L^\infty(I)$: If $\varphi_n \in C(I)$, and $\varphi_n \to f$ in the supnorm, then $(\varphi_n)$ is a Cauchy sequence with respect to the uniform norm (when applied to continuous functions, the uniform norm and the $L^\infty$ norms are identical). Therefore, there exists a continuous function $\varphi$ such that $\varphi_n \to \varphi$ uniformly. Then $f(x) = \varphi(x)$ almost everywhere. But not every equivalence class function in $L^\infty$ has a continuous function in it (for instance $f = \chi_{[0,1/2]}$).

**Problem 12.14:** Let $p$ and $q$ be such that $1 \leq p < q \leq \infty$.

First we construct a function $f \in L^p \backslash L^q$. Let $\alpha$ be a non-negative number and set $f(x) = x^{-\alpha}\chi_{[0,1]}$. Then

$$||f||_p^p = \int_0^1 x^{-\alpha\, p}\, dx,$$

which is finite if $\alpha p < 1$. Moreover

$$||f||_q^q = \int_0^1 x^{-\alpha\, q}\, dx$$

which is infinite if $\alpha q > 1$. Consequently, $f \in L^p \backslash L^q$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

To construct a function $f \in L^q \backslash L^p$, set $f = x^{-\alpha}\chi_{[1,\infty)}$. Then

$$||f||_p^p = \int_1^\infty x^{-\alpha p}\, dx$$

which is infinite if $\alpha p < 1$. Moreover

$$||f||_q^q = \int_1^\infty x^{-\alpha q}\, dx$$

which is finite if $\alpha q > 1$. Thus, $f \in L^1 \backslash L^p$ if

$$\frac{1}{q} < \alpha < \frac{1}{p}.$$

(The arguments above need slight modifications if $q = \infty$, but the idea is the same.)

Consider the function

$$f(x) = \frac{1}{\left(|x|\,(1 + \log^2|x|)\right)^{1/2}}.$$

That $f \in L^2$ is clear, since

$$||f||_2^2 = \int_{-\infty}^\infty \frac{1}{|x|(1 + \log^2|x|)}\, dx = 2\int_0^\infty \frac{1}{x(1 + \log^2 x)}\, dx = \{x = e^t\}$$

$$2\int_{-\infty}^\infty \frac{1}{e^t(1 + t^2)} e^t\, dt = 2\pi.$$

Moreover, if $p > 2$, then note that there exists a $\delta > 0$ such that

$$x^{(p-2)/2}(1+\log^2 x)^{p/2} \le 1$$

when $x \in (0,\, \delta)$. Then

$$||f||_p^p \ge \int_0^\delta \frac{1}{x^{p/2}(1+\log^2 x)^{p/2}}\, dx = \int_0^\delta \frac{1}{x} \underbrace{\frac{1}{x^{(p-2)/2}\,(1+\log^2 x)^{p/2}}}_{\ge 1}\, dx = \infty.$$

Analogously, if $p < 2$, then there exists an $M$ such that

$$x^{(p-2)/2}(1+\log^2 x)^{p/2} \le 1$$

when $x \ge M$. Then

$$||f||_p^p \ge \int_M^\infty \frac{1}{x^{p/2}(1+\log^2 x)^{p/2}}\, dx = \int_M^\infty \frac{1}{x} \underbrace{\frac{1}{x^{(p-2)/2}\,(1+\log^2 x)^{p/2}}}_{\ge 1}\, dx = \infty.$$

**Problem 12.15:** Let $\alpha \in (0, 1)$, and let $m, n \in (1, \infty)$ be such that $1/m + 1/n = 1$ (we will determine suitable values for $\alpha, m, n$ later). Then from Hölder's inequality we obtain

$$(1) \qquad ||f||_r^r = \int |f|^r = \int |f|^{\alpha r} |f|^{(1-\alpha) r} \le \left(\int |f|^{\alpha m r}\right)^{1/m} \left(\int |r|^{(1-\alpha) n r}\right)^{1/n}.$$

In order to obtain the desired right hand side, we must pick $\alpha, m, n$ so that

$$\begin{aligned} \alpha m r =& p, \\ (1-\alpha) n r =& q, \\ (1/m) + (1/n) =& 1. \end{aligned}$$

To obtain an equation for $\alpha$, we eliminate $m$ and $n$:

$$\frac{(1-\alpha) r}{q} = \frac{1}{n} = 1 - \frac{1}{m} = 1 - \frac{\alpha r}{p}.$$

Solving for $\alpha$ we obtain

$$\alpha = \frac{pq - pr}{rq - rp} = \frac{1/r - 1/q}{1/p - 1/q}.$$

Equation (1) now takes the form

$$||f||_r \le \left( \left(||f||_p^p\right)^{1/m} \left(||f||_q^q\right)^{1/n} \right)^{1/r} = ||f||_p^{p/mr}\, ||f||_q^{q/nr}.$$

Finally note that

$$\begin{aligned} \frac{p}{mr} =& \alpha = \frac{1/r - 1/q}{1/p - 1/q}, \\ \frac{q}{nr} =& 1 - \alpha = 1 - \frac{1/r - 1/q}{1/p - 1/q} = \frac{1/p - 1/r}{1/p - 1/q}. \end{aligned}$$

**Problem 1:** Let $\lambda$ be a real number such that $\lambda \in (0,1)$, and let $a$ and $b$ be two non-negative real numbers. Prove that

$$a^\lambda\, b^{1-\lambda} \leq \lambda\, a + (1-\lambda)\, b, \tag{2}$$

with equality iff $a = b$.

*Solution:* For $b = 0$ equation (2) reduces to $0 \leq \lambda\, a$ which is clearly true.

When $b \neq 0$ we divide (2) by $b$ and set $t = a/b$ to obtain

$$t^\lambda \leq \lambda\, t + 1 - \lambda.$$

Set

$$f(t) = \lambda\, t + 1 - \lambda - t^\lambda.$$

We need to prove that $f(t) \geq 0$ when $t \geq 0$. First note that $f(0) = 1 - \lambda > 0$ and that $\lim_{t\to\infty} f(t) = \infty$. Since $f$ is differentiable, we therefore need only investigate the points where $f'(t) = 0$. We find

$$f'(t) = \lambda - \lambda\, t^{\lambda-1}$$

so $f'(t) = 0$ happens only when $t = 1$. Now $f(1) = 0$ so it follows that $f(t) \geq 0$ for all $t \geq 0$, and that $f(t) = 0$ iff $t = 1$ (which is to say $a = b$).

**Problem 2:** [Hölder's inequality] Suppose that $p$ is a real number such that $1 < p < \infty$, and let $q$ be such that $p^{-1} + q^{-1} = 1$. Let $(X, \mu)$ be a measure space, and suppose that $f \in L^P(X, \mu)$ and $g \in L^q(X, \mu)$. Prove that $fg \in L^1(X, \mu)$, and that

$$||fg||_1 \leq ||f||_p\, ||g||_q. \tag{3}$$

Prove that equality holds iff $\alpha|f|^p = \beta|g|^q$ for some $\alpha, \beta$ such that $\alpha\beta \neq 1$.

*Solution:* Suppose $||f||_p = 0$, then $f = 0$ a.e. and so (3) holds since both sides are identically zero. Analogously, (3) holds when $||g||_q = 0$.

Now suppose $||f||_p \neq 0$ and $||g||_q \neq 0$. Set

$$a = \left|\frac{f(x)}{||f||_p}\right|^p, \qquad b = \left|\frac{g(x)}{||g||_q}\right|^q, \qquad \lambda = \frac{1}{p}.$$

Then invoke (2), observing that $q(1-\lambda) = q(1-1/p) = q\,(1/q) = 1$, to obtain

$$\frac{|f(x)|}{||f||_p}\,\frac{|g(x)|}{||g||_q} \leq \frac{1}{p}\,\frac{|f(x)|^p}{||f||_p^p} + \left(1 - \frac{1}{p}\right)\,\frac{|g(x)|^q}{||g||_q^q}.$$

Integrate over $X$ to obtain

$$\frac{1}{||f||_p\,||g||_q}\int_X |f(x)|\,|g(x)|\,d\mu(x) \leq \underbrace{\frac{1}{p}\,\frac{||f||_p^p}{||f||_p^p} + \left(1 - \frac{1}{p}\right)\,\frac{||g||_q^q}{||g||_q^q}}_{=1}.$$

Multiply by $||f||_p\,||g||_q$ to obtain (3).

**Problem 3:** [Minkowski's inequality] Let $(X, \mu)$ be a measure space, and let $p$ be a real number such that $1 \leq p \leq \infty$. Prove that for $f, g \in L^p(X, \mu)$,

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

*Solution:*

Suppose $p = 1$:

$$||f + g||_1 = \int |f(x) + g(x)| \leq \int \big(|f(x)| + |g(x)|\big) = \int |f(x)| + \int |g(x)| = ||f||_1 + ||g||_1.$$

Suppose $p = \infty$:

$$\begin{aligned} ||f + g||_\infty = \operatorname{ess\,sup} |f(x) + g(x)| &\leq \operatorname{ess\,sup}\big(|f(x)| + |g(x)|\big) \\ &\leq \operatorname{ess\,sup} |f(x)| + \operatorname{ess\,sup} |g(x)| = ||f||_\infty + ||g||_\infty. \end{aligned}$$

Suppose $p \in (1, \infty)$: The triangle inequality yields

$$|f(x) + g(x)|^p = |f(x) + g(x)|\,|f(x) + g(x)|^{p-1} \leq \big(|f(x)| + |g(x)|\big)\,|f(x) + g(x)|^{p-1}.$$

Integrate both sides:

$$||f + g||_p^p \leq \int |f(x)|\,|f(x) + g(x)|^{p-1} + \int |g(x)|\,|f(x) + g(x)|^{p-1}.$$

Now apply Hölder:

$$||f + g||_p^p \leq ||f||_p\,||\,|f + g|^{p-1}||_q + ||g||_p\,||\,|f + g|^{p-1}||_q = \big(||f||_p + ||g||_p\big)\left(\int |f(x) + g(x)|^{q(p-1)}\right)^{1/q}.$$

Now use that $q = 1/(1 - 1/p) = p/(p-1)$ to see that $q(p-1) = p$ to get

$$||f + g||_p^p \leq \big(||f||_p + ||g||_p\big)\left(\int |f(x) + g(x)|^p\right)^{1/q} = \big(||f||_p + ||g||_p\big)\,||f + g||_p^{p/q}.$$

Observe that $p/q = p(1 - 1/p) = p - 1$ to obtain

$$||f + g||_p^p \leq \big(||f||_p + ||g||_p\big)\,||f + g||_p^{p-1}$$

which gives Minkowski upon division by $||f + g||_p^{p-1}$.