

Homework set 12 — APPM5450, Spring 2011 — Solutions

Problem 12.2: We use “ \uplus ” to denote disjoint unions.

(a) Suppose that $A, B \in \mathcal{A}$. Then note that $A \setminus B = A \cap B^c = (A^c \cup B)^c$. It now follows directly from the axioms that $A \setminus B \in \mathcal{A}$.

(b) Set $C = B \setminus A$. Then $B = A \uplus C$, so

$$\mu(B) = \mu(A \uplus C) = \mu(A) + \mu(C) \geq \mu(A).$$

(c) Set $C = A \cap B$. Then $A = (A \setminus B) \uplus C$ and $B = (B \setminus A) \uplus C$ so

$$\begin{aligned} \mu(A \cup B) &= \mu((A \setminus B) \uplus C \uplus (B \setminus A)) = \mu(A \setminus B) + \mu(C) + \mu(B \setminus A) \\ &\leq \mu(A \setminus B) + \mu(C) + \mu(C) + \mu(B \setminus A) \\ &= \mu((A \setminus B) \uplus C) + \mu(C \uplus (B \setminus A)) = \mu(A) + \mu(B). \end{aligned}$$

Problem 12.3: The trick is to write $\bigcup_{n=1}^{\infty} A_n$ as a disjoint union. For $n = 1, 2, 3, \dots$ set $B_n = A_{n+1} \setminus A_n$. Then

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup \left(\bigcup_{n=1}^{\infty} B_n \right),$$

where the union on the right is a disjoint one. Now use additivity twice:

$$\begin{aligned} \mu \left(\bigcup_{n=1}^{\infty} A_n \right) &= \mu \left(A_1 \cup \left(\bigcup_{n=1}^{\infty} B_n \right) \right) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N \rightarrow \infty} \left(\mu(A_1) + \sum_{n=1}^N \mu(B_n) \right) = \lim_{N \rightarrow \infty} \mu \left(A_1 \cup \left(\bigcup_{n=1}^N B_n \right) \right) = \lim_{N \rightarrow \infty} \mu(A_N). \end{aligned}$$

For the second part, set $C = \bigcap_{n=1}^{\infty} A_n$ and $C_n = A_n \setminus A_{n+1}$. Then

$$\mu(A_N) = \mu \left(C \cup \left(\bigcup_{n=N}^{\infty} C_n \right) \right) = \mu(C) + \sum_{n=N}^{\infty} \mu(C_n).$$

Since $\infty > \mu(A_1) \geq \sum_{n=1}^{\infty} \mu(C_n)$, we find that

$$\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} \mu(C_n) = 0,$$

which completes the proof. For the counterexample, consider $X = \mathbb{R}^2$, and $A_n = \{x = (x_1, x_2) : |x_2| < 1/n\}$. Then $\mu(A_n) = \infty$ for all n , but $\bigcap_{n=1}^{\infty} A_n$ is the x_1 -axis, which has measure zero.

Problem 12.5: Straight-forward.

Problem 12.7:

Reflexivity: It is obvious that $f(x) = f(x)$ a.e.

Symmetry: If $f(x) = g(x)$ a.e., then obviously $g(x) = f(x)$ a.e.

Transitivity: Suppose that $f(x) = g(x)$ a.e. and that $g(x) = h(x)$ a.e. Set

$$A = \{x : f(x) \neq g(x)\}$$

$$B = \{x : g(x) \neq h(x)\}$$

$$C = \{x : f(x) \neq h(x)\}.$$

We know that $\mu(A) = \mu(B) = 0$, and we want to prove that $\mu(C) = 0$. It is clearly the case that $C \subseteq A \cup B$, and then it follows directly that $\mu(C) \leq \mu(A) + \mu(B) = 0$.