

## Homework set 10 — APPM5450, Spring 2011

From the textbook: 11.18, 11.13, 11.16.

In 11.16, you're free to assume that  $f$  is smooth (or that  $f \in \mathcal{S}(\mathbb{R}^3)$ ), if you like. You may also assume that  $f \in L^1$  in 11.18, but please return to the problem once we've described the action of  $\mathcal{F}$  on  $L^2$ .

**Problem 1:** Let  $R$  denote a real number such that  $0 < R < \infty$  and define

$$f_n(x) = \begin{cases} n \cos(nx) & \text{for } |x| \leq R, \\ 0, & \text{for } |x| > R. \end{cases}$$

For which numbers  $R$ , if any, is it the case that  $f_n \rightarrow 0$  in  $\mathcal{S}^*$ ?

**Problem 2:** (Optional review of old material.) Prove that  $C_c(\mathbb{R}^d)$  is dense in  $C_0(\mathbb{R}^d)$ . Prove that  $C_0(\mathbb{R}^d)$  is a closed subset of  $C_b(\mathbb{R}^d)$ .

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*What follows is a set of review questions for Chapter 11. They are not part of the home work but you may find them useful in preparing for the third midterm and the final:*

What does it mean for  $\varphi_n \rightarrow \varphi$  in  $\mathcal{S}$ ?

Make absolutely sure that you understand problems like 11.4, 11.10a.

Prove that if  $\varphi_n \rightarrow \varphi$  in  $\mathcal{S}$ , then  $x\varphi_n(x) \rightarrow x\varphi(x)$  and  $\partial\varphi_n \rightarrow \partial\varphi$  in  $\mathcal{S}$ .

Let  $T$  be a linear map from  $\mathcal{S}$  to  $\mathbb{R}$ . What does it mean for  $T$  to be continuous? Prove that if there exists a finite  $C$  and a finite  $N$  such that  $|T(\varphi)| \leq C \sum_{|\alpha|, n \leq N} \|\varphi\|_{\alpha, n}$ , then  $T$  is continuous.

Let  $T \in \mathcal{S}^*(\mathbb{R}^d)$ , and let  $\alpha$  be a multi-index. Define  $x^\alpha T$ . Prove that what you define is a tempered distribution.

Prove that  $n^2 \sin(nx) \rightarrow 0$  in  $\mathcal{S}^*$ .

Is the Schwartz space dense in  $\mathcal{S}^*$ ?

Prove that  $\sup_x |x^\beta \partial^\alpha \varphi(x)| < \infty \forall \alpha, \beta \iff \sup_x |(1 + |x|^2)^{k/2} \partial^\alpha \varphi(x)| < \infty \forall \alpha, k$ .

Assume that  $\int |f|^2 < \infty$ , set  $\langle T, \varphi \rangle = \int f \varphi$ . Prove that  $T \in \mathcal{S}^*$ .

Let  $H$  be a function such that  $H(x) = 1$  if  $x \geq 0$ , zero otherwise. Prove that  $H \in \mathcal{S}^*$ . Calculate  $H'$ . Let  $H_R$  denote the function that is 1 when  $0 \leq x \leq R$  and zero otherwise. Prove that  $H_R \rightarrow H$  in  $\mathcal{S}^*$  as  $R \rightarrow \infty$ .

Let  $\psi$  be a Schwartz function such that  $\int \psi = 0$ . Set  $\varphi_n(x) = n\psi(nx)$ . Does  $\varphi_n$  converge in  $\mathcal{S}$ ? Does  $\varphi_n$  converge in  $\mathcal{S}^*$ ?

Prove that  $\text{PV}(1/x)$  is a continuous functional on  $\mathcal{S}$ .

What is the distributional derivative of  $\text{PV}(1/x)$ ?

Define  $\hat{T}$  for  $T \in \mathcal{S}^*$ . Prove that what you define is a continuous map on  $\mathcal{S}$ .