Applied Analysis (APPM 5450): Final exam

1.30pm – 4.00pm, May 3, 2011. Closed books.

Problem 1: (14p) Let d be a positive integer denoting dimension.

(a) (4p) State the definition of the Sobolev space $H^s(\mathbb{R}^d)$ for $s \geq 0$.

(b) (4p) State the Riemann-Lebesgue lemma. (You do not need to give the proof.)

(c) (6p) Prove that if s is large enough (how large?), then $H^s(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$.

Problem 2: (26p) Consider the Hilbert spaces $H_1 = \ell^2(\mathbb{Z})$ and $H_2 = L^2(\mathbb{R})$, and define operators $A_1 \in \mathcal{B}(H_1)$ and $A_2 \in \mathcal{B}(H_2)$ via

$$\begin{aligned} [A_1 u](n) &= \arctan(n)\, u(n) \qquad n \in \mathbb{Z}, \\ [A_2 u](x) &= \arctan(x)\, u(x) \qquad x \in \mathbb{R}. \end{aligned}$$

(a) (7p) Is A_1 compact? Self-adjoint? Unitary? One-to-one? Does it have closed range? Please motivate your answers briefly.

(b) (6p) Specify $\sigma(A_1)$, $\sigma_{\rm p}(A_1)$, $\sigma_{\rm c}(A_1)$, $\sigma_{\rm r}(A_1)$, and $||A_1||$. No motivation required.

(c) (7p) Is A_2 compact? Self-adjoint? Unitary? One-to-one? Does it have closed range? Please motivate your answers briefly.

(d) (6p) Specify $\sigma(A_2)$, $\sigma_{\rm p}(A_2)$, $\sigma_{\rm c}(A_2)$, $\sigma_{\rm r}(A_2)$, and $||A_2||$. No motivation required.

Problem 3: (14p) Define for $x \in \mathbb{R}$ and $n = 1, 2, 3, \ldots$ the functions

$$T_n(x) = \frac{n\,x}{n\,x^2 + 1}.$$

Does $(T_n)_{n=1}^{\infty}$ converge in $\mathcal{S}^*(\mathbb{R})$? If so, to what? Please motivate your answer.

Problem 4: (22p) Consider the Banach space $X = L^5(\mathbb{R})$ equipped with the standard norm

$$||f||_5 = \left(\int_{\mathbb{R}} |f(x)|^5 \, dx \right)^{1/5}.$$

(a) (6p) What is X^*? Describe the action of an element of X^*.

(b) (6p) Which of the following statements are necessarily true (no motivation required):
  (i) Any bounded sequence in X has a weakly convergent subsequence.
  (ii) The weak-$\star$ topology on X is identical to the weak topology.
  (iii) Any bounded set $\Omega \subseteq X$ is pre-compact in the weak topology.
  (iv) Any bounded set $\Omega \subseteq X$ that is closed in the norm topology is compact in the weak topology.

(c) (10p) Let α be a real number and define the functions $(f_n)_{n=1}^{\infty}$ via

$$f_n(x) = n^{\alpha} \, \chi_{[n,\, n+1/n]}(x) = \begin{cases} 0 & x < n \\ n^{\alpha} & n \leq x \leq n + 1/n \\ 0 & n + 1/n < x. \end{cases}$$

For which α does $(f_n)_{n=1}^{\infty}$ converge in norm? Weakly? Motivate your answer carefully.

Problem 5: (24p) Let h and g be measurable functions on $\mathbb{R}$, and let $(h_n)_{n=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$. Suppose that $h_n \, g \in L^1(\mathbb{R})$ for all n, and that

$$\lim_{n \to \infty} h_n(x) = h(x) \qquad \text{for every } x \in \mathbb{R}.$$

Please answer the following questions, and provide brief motivations:

(a) (8p) Suppose that h_n and g are non-negative and that $\int h_n \, g = 1/n$.
Is it necessarily the case that $\int h \, g = 0$?

(b) (8p) Suppose that $|h_n(x)| \leq |h(x)|$ for all x and n, that $h \in L^2(\mathbb{R})$, and that $g \in L^2(\mathbb{R})$.
Is it necessarily the case that $\lim_{n \to \infty} \int h_n \, g = \int h \, g$?

(c) (8p) Suppose that $0 \leq h_1(x) \leq h_2(x) \leq h_3(x) \leq \cdots$ for all x and set $c_n = \int h_n \, g$.
Is the sequence $(c_n)_{n=1}^{\infty}$ necessarily convergent?
(And yes, if $c_n \to \infty$ or $c_n \to -\infty$, we do say that (c_n) is convergent.)