11.5: Note that

$$\frac{1}{x+i\varepsilon} = \frac{x}{\varepsilon^2+x^2} - i\frac{\varepsilon}{\varepsilon^2+x^2}.$$

Fix a $\varphi \in \mathcal{S}$. You need to prove that

$$\lim_{\varepsilon\to 0}\langle i\frac{\varepsilon}{\varepsilon^2+x^2},\,\varphi\rangle \to -i\pi\varphi(0). \tag{1}$$

and that

$$\lim_{\varepsilon\to 0}\langle \frac{x}{\varepsilon^2+x^2},\,\varphi\rangle \to \langle \mathrm{PV}\left(\frac{1}{x}\right),\,\varphi\rangle, \tag{2}$$

Proving (1) is simple:

$$\langle i\frac{\varepsilon}{\varepsilon^2+x^2},\,\varphi\rangle = \int_{-\infty}^{\infty} i\frac{\varepsilon}{\varepsilon^2+x^2}\,\varphi(x)\,dx = \{\text{Set } x = \varepsilon y\} = \cdots$$

For (2) we need to work a bit more (unless I overlook a simpler solution)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\langle \frac{x}{\varepsilon^2+x^2},\,\varphi\rangle - \langle \mathrm{PV}\left(\frac{1}{x}\right),\,\varphi\rangle \\
&\qquad = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{x}{\varepsilon^2+x^2}\,\varphi(x)\,dx - \lim_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\frac{1}{x}\,\varphi(x)\,dx \\
&\qquad = \underbrace{\lim_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\left(\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right)\,\varphi(x)\,dx}_{=S_1} + \underbrace{\lim_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}}\frac{x}{\varepsilon^2+x^2}\,\varphi(x)\,dx}_{=S_2}.
\end{aligned}$$

First we bound $|S_1|$. Note that when $|x| \geq \sqrt{\varepsilon}$, we have

$$\left|\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right| = \frac{\varepsilon^2}{|x|(\varepsilon^2+x^2)} \leq \frac{\varepsilon^2}{|x|^3} \leq \frac{\varepsilon^2}{\varepsilon^{3/2}} = \sqrt{\varepsilon}.$$

Consequently,

$$\begin{aligned}
|S_1| &\leq \limsup_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\left|\frac{x}{\varepsilon^2+x^2} - \frac{1}{x}\right|\,|\varphi(x)|\,dx \\
&\leq \limsup_{\varepsilon\to 0}\int_{|x|\geq\sqrt{\varepsilon}}\sqrt{\varepsilon}\frac{1}{(1+|x|^2)}\,\underbrace{|(1+|x|^2)\varphi(x)|}_{\leq||\varphi||_{0,2}}\,dx = 0.
\end{aligned}$$

In bounding S_2 we use that

$$\int_{|x|\leq\sqrt{\varepsilon}}\frac{x}{\varepsilon^2+x^2}\,\varphi(0)\,dx = 0,$$

and that

$$|\varphi(x) - \varphi(0)| \leq |x|\,||\varphi'||_{\mathrm{u}} \leq |x|||\varphi||_{1,0},$$

to obtain

$$\begin{aligned}
|S_2| &= \left|\lim_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}}\frac{x}{\varepsilon^2+x^2}\,(\varphi(x)-\varphi(0))\,dx\right| \\
&\leq \limsup_{\varepsilon\to 0}\int_{|x|\leq\sqrt{\varepsilon}}\underbrace{\frac{|x|}{\varepsilon^2+x^2}\,|x|}_{\leq 1}\,||\varphi||_{1,0}\,dx = 0.
\end{aligned}$$