Review of compact sets & compact operators on H.S.

Recall that if Ω is a subset of a H.S., then TFAE:

(a) Ω is compact
(b) Every open cover of Ω has a finite subcover
(c) Every sequence in Ω has a convergent subseq
(d) Ω is closed and totally bdd (i.e. $\forall \varepsilon > 0$, $\exists (x_j)_{j=1}^{J}$ s.t. $\Omega \subseteq \bigcup_{j=1}^{J} B_\varepsilon(x_j)$)

Recall that if $A \in \mathcal{B}(H)$, then TFAE:

(1) A is compact
(2) For any bdd set Ω, $A\Omega$ is precompact
(3) If (x_n) is a bdd seq, then (Ax_n) has a convergent subseq
(4) If $x_n \to x$ weakly, then $Ax_n \to Ax$ in norm
(5) For any $\varepsilon > 0$, $\exists$ a finite rank operator A_N s.t. $\|A - A_N\| < \varepsilon$

We will next show that in a H.S, we can add a property (e) to the list above that is the set equivalent of condition (5).

Proof that $(3) \Leftrightarrow (4)$:

$(3) \Rightarrow (4)$ Assume that ~~(3) holds and that~~ $x_n \to x$. but that $Ax_n \not\to Ax$.
Then $\forall y$: $(Ax - Ax_n, y) = (x - x_n, A^* y) \to 0$ so $Ax_n \rightharpoonup Ax$.
~~Then~~ Moreover $\exists$ $\varepsilon > 0$ & $(n_j)_{j=1}^{\infty}$ s.t. $\|Ax_{n_j} - Ax\| > \varepsilon$ $\forall j$.
~~But then~~ Note that (x_{n_j}) is a bdd seq, but no subseq of (Ax_{n_j}) may converge strongly since $Ax_{n_j} \rightharpoonup Ax$ but $\|Ax_{n_j} - Ax\| > \varepsilon$
Thus (3) cannot hold.

$(4) \Rightarrow (3)$ Assume that (4) holds, and that (x_n) is a bdd seq.
Banach-Alaoglu $\Rightarrow$ $\exists$ (n_j) s.t. $x_{n_j} \rightharpoonup x$ for some x.
Then (4) $\Rightarrow$ $Ax_{n_j} \to Ax$ which proves (3).

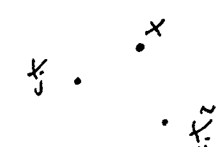
Thm Let Ω be a subset of an inf-dim. seperable H.S. H.

(a) If Ω is pre-compact, and $(\varphi_n)_{n=1}^{\infty}$ is an ON-basis for H then for any $\varepsilon>0$, $\exists N$ s.t. $\|(I-P_N)x\|<\varepsilon \quad \forall x\in\Omega$, where $P_N=$ orthog proj^n onto span$(\varphi_1,\dots,\varphi_N)$.

(b) If Ω is a bdd set, and $(\varphi_n)_{n=1}^{\infty}$ is an ON-basis for H, such that $\forall\varepsilon>0$, $\exists N$ s.t. $\|(I-P_N)x\|<\varepsilon \quad \forall x\in\Omega$, where P_N is as in (a), then Ω is pre-compact.

Note: As an implication of (a), if Ω is pre-compact, and $(\varphi_n)_{n=1}^{\infty}$ is any ON-set in H, then for any $\varepsilon>0$, $\exists N$ s.t. $\sum_{n=N+1}^{\infty}|(\varphi_n,x)|^2<\varepsilon^2$.
To see this simply apply the thm to the proj^n of H onto span$(\varphi_n)_{n=1}^{\infty}$.

Proof (a) Suppose Ω is precompact and let $(\varphi_n)_{n=1}^{\infty}$ be an ON-basis for H.
Given $\varepsilon>0$, pick $(x_j)_{j=1}^{J}$ s.t. $\Omega\subseteq\bigcup_{j=1}^{J}B_{\varepsilon/2}(x_j)$.
For each j, pick N_j s.t. $\|(I-P_{N_j})x_j\|<\frac{\varepsilon}{2}$ (possible since $\|(I-P_N)x_j\|\to 0$ as $N\to\infty$)
Set $\tilde{x}_j=P_{N_j}x_j$.
Set $N=\max_{1\le j\le J} N_j$.
Then given $x\in\Omega$, pick x_j s.t. $x\in B_{\varepsilon/2}(x_j)$



then $\|(I-P_N)x\|=\|(I-P_N)(\tilde{x}_j+x-\tilde{x}_j)\|=\underbrace{\|(I-P_N)\tilde{x}_j\|}_{=0}+\|(I-P_N)(x-\tilde{x}_j)\|\le$
$\le\|x-\tilde{x}_j\|<\|x-x_j\|+\|x_j-\tilde{x}_j\|<\varepsilon$.

(b) Suppose that the conditions given in (b) hold.
Set $C=\sup\{\|x\|:x\in\Omega\}$.
Given $\varepsilon>0$, pick N s.t. $\|(I-P_N)x\|<\varepsilon/2 \quad \forall x\in\Omega$.
Set $R_N=\{x\in\text{span}(\varphi_1,\dots,\varphi_N):\|x\|\le C\}$
R_N is compact $\Rightarrow \exists (x_j)_{j=1}^{J}$ s.t. $R_N\subseteq\bigcup_{j=1}^{J}B_{\varepsilon/2}(x_j)$
~~Given $x\in\Omega$~~ Then $(B_\varepsilon(x_j))_{j=1}^{J}$ is an open ε-cover of Ω since for any $x\in\Omega$, set $x=P_Nx+(I-P_N)x$. Pick j s.t. $P_Nx\in B_{\varepsilon/2}(x_j)$
Then $\|x-x_j\|=\|P_Nx-x_j+(I-P_N)x\|\le\|P_Nx-x_j\|+\|(I-P_N)x\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

Example Consider $H = L^2(\mathbb{T})$ and the set $\Omega = \{ f \in H : \|f\|_u \le c_1 \ \& \ \cancel{\|f'\| \le c_1}\ \cancel{\sup|f'(x)| \le c_1}\ \|f'\|_u \le c_1 \}$

Set $\varphi_n = \frac{e^{inx}}{\sqrt{2\pi}}$ and P_N = orthog proj^n onto span$(\varphi_{-N}, \dots, \varphi_N)$

~~We will show that~~ $\cancel{\|(I-P_N)f\|^2 = \sum_{|n|\ge N} |(\varphi_n, f)|^2 \to 0 \text{ as } N \to \infty}$

We have for $n \neq 0$

$$|(\varphi_n, f)| = \left| \int_{-\pi}^{\pi} \frac{e^{-inx}}{\sqrt{2\pi}} f(x)\,dx \right| = \left| \left[\frac{e^{-inx}}{-in\sqrt{2\pi}} f(x) \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{e^{-inx}}{-in\sqrt{2\pi}} f'(x)\,dx \right| \le$$

$$\le \frac{2c_1}{n\sqrt{2\pi}} + \frac{c_1 2\pi}{n\sqrt{2\pi}} \le \frac{C_1}{n}$$ for some C_1 that does not depend on n or f

Thus $$\|(I-P_N)f\|^2 = \sum_{|n|\ge N} |(\varphi_n, f)|^2 \le \sum_{|n| \ge N} \frac{C_1^2}{n^2} \le \frac{C_2}{N}$$

So for any $\varepsilon > 0$, we can find N s.t. $\|(I-P_N)f\| \le \varepsilon$.

Thus Ω is ~~compa~~ pre-compact.

Hilbert-Schmidt operators (for orientation only)

Def^n Let H be a H.S. and let $A \in \mathcal{B}(H)$.

We say that A is a Hilbert-Schmidt operator if for some ON-basis $(\varphi_n)_{n=1}^{\infty}$, it is the case that $\sum_{n=1}^{\infty} \|A\varphi_n\|^2 < \infty$.

Claim Every H-S operator is compact (but not every compact op is H-S.)

Example Let A be a compact S-A op.

Spectral thm $\Rightarrow$ $\exists$ ON-basis $(\varphi_n)_{n=1}^{\infty}$ & evals $(\lambda_n)_{n=1}^{\infty}$ s.t. ~~$Au = \sum$~~ $A\varphi_n = \lambda_n \varphi_n$.

Then $\sum_{n=1}^{\infty} \|A\varphi_n\|^2 = \sum_{n=1}^{\infty} |\lambda_n|^2$ so A is H-S if $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$.

Is this a necessary cond^n in addition to sufficient?

Could it be the case that $\sum_{n=1}^{\infty} |\lambda_n|^2 = \infty$ but $\sum_{n=1}^{\infty} \|A\varphi_n\|^2 < \infty$ for some other basis?

No:

Claim If A is H-S, and if (φ_n) & (ψ_n) are two ON-bases, then $\sum \|A\varphi_n\|^2 = \sum \|A\psi_n\|^2$.

Moreover, $\|A\|_{HS} = \left(\sum_{n=1}^{\infty} \|A\varphi_n\|^2 \right)^{1/2}$ is a norm on the set of H-S ops.

Example $H = \mathbb{C}^n$ $A \in \mathcal{B}(H)$ $A = \begin{bmatrix} a_{11} & - & a_{1n} \\ & & | \\ & & a_{nn} \end{bmatrix}$

$\|A\|_{HS} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \sqrt{\text{trace}(AA^*)}$ ← also known as the Frobenius norm.

Example $H = L^2(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$

$[Au](x) = \int_\Omega k(x,y) u(y) dy$

$\|A\|_{HS}^2 = \int_\Omega \int_\Omega |k(x,y)|^2 dx dy$ so A is H-S $\Leftrightarrow$ $k \in L^2(\Omega \times \Omega)$.

If A is also S-A, then $\exists$ (φ_n) & (λ_n) s.t.

$[Au](x) = \sum_n \lambda_n \varphi_n (\varphi_n, u) = \sum_n \lambda_n \varphi_n(x) \int_\Omega \overline{\varphi_n(y)} u(y) dy = \int_\Omega \left(\sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \overline{\varphi_n(y)} \right) u(y) dy$

So when A is H-S, $k(x,y) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \overline{\varphi_n(y)}$

Functions of operators (orientation only)

Let $f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_d z^d$ be a polynomial.

Natural def^n: $f(A) = \alpha_0 + \alpha_1 A + \cdots + \alpha_d A^d$

If $A = \sum_{n=1}^{\infty} \lambda_n P_n$, then $A^k = \sum_{n=1}^{\infty} \lambda_n^k P_n$ since $P_n^k = P_n$ & $P_n P_m = 0$ if $n \neq m$.

Thus $f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n$

Now let f be a function from $\mathbb{C}$ to $\mathbb{C}$.

If $f(z)=\sum_{n=0}^{\infty} \alpha_n z^n$ for $|z|<R$ ← analytic on $B(R)$,

and if $\|A\|<R$, then we may define $f(A)=\sum_{n=0}^{\infty} \alpha_n A^n$ ← ~~absolutely conv~~ convergent in norm.

<u>Example</u> $f(z)=(1-z)^{-1}=\sum_{n=0}^{\infty} z^n$ for $|z|<1$

$f(A)=(I-A)^{-1}=\sum_{n=0}^{\infty} A^n$ as long as $\|A\|<1$.

<u>Example</u> $f(z)=e^z=\sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all $z\in\mathbb{C}$.

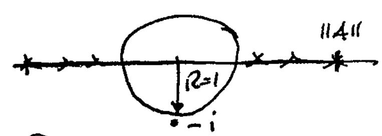
$f(A)=e^A=\sum_{n=0}^{\infty} \frac{A^n}{n!}$ $\forall A\in B(H)$

If $\cancel{\rho(A)}\ A=\sum_{n=1}^{\infty} \lambda_n P_n$ & $\cancel{\rho(A)}\ r(A)=\sup|\lambda_n|<R$, then $f(A)=\sum_{n=1}^{\infty} f(\lambda_n) P_n$ $(*)$

Actually, we can use $(*)$ as a def^n, even if $\cancel{\rho(A)}\ r(A)>R$,

the only condition is that f is continuous on an open set $\Omega\supseteq\sigma(A)$.

<u>Example</u> $f(z)=\frac{z-i}{z+i}$



$A=\sum_{n=1}^{\infty}\lambda_n P_n$

We cannot define $f(A)$ via powerseries.

However, we can define $f(A)=\sum \frac{\lambda_n-i}{\lambda_n+i} P_n$

If A is S-A, then $\lambda_n\in\mathbb{R}$ $\Rightarrow \left|\frac{\lambda_n-i}{\lambda_n+i}\right|=1$ $\Rightarrow \|f(A)x\|=\|x\|$ $\forall x$

so $f(A)$ is unitary.

Note that* $f(\sigma(A))=\sigma(f(A))$. Cayley transform

In general, if $f(\sigma(A))\subseteq$ $\{z: |z|=1\}$, then $f(A)$ is unitary

$\{z: z\in\mathbb{R}\}$, then $f(A)$ is S-A

$\{z: \cancel{\ \ }\ \mathrm{Re}\, z=0\}$, then $f(A)$ is skew-sym.

$f(A)$ is always normal.

More generally, if $A=\int_{\sigma(A)}\lambda\, dP(\lambda)$, then $f(A)=\int_{\sigma(A)} f(\lambda)\, dP(\lambda)$