Applied Analysis (APPM 5450): Midterm 3
8.30am – 9.50pm, April 19, 2010. Closed books.

Note: *You may want to save problems marked with a star for last.*

Problem 1: (15 points) Let $g, h \in L^2(\mathbb{R})$ and set $f = g * h$. Prove that $||f||_{\text{u}} \leq ||g||_{L^2}\, ||h||_{L^2}$ (where $||f||_{\text{u}} = \sup_x |f(x)|$). Is it necessarily the case that $f \in C_0(\mathbb{R})$? Motivate your answer briefly.

Problem 2: (26 points) In this problem, $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is the Schwartz space over the real line, a is a non-zero real number, and $\mathcal{F}$ is the Fourier transform.

(a) [6p] Define the operator $D_a : \mathcal{S} \to \mathcal{S}$ via $[D_a\varphi](x) = \varphi(a\,x)$. Show that for some $b, c \in \mathbb{R}$

(EQ1) $$\mathcal{F}\, D_a\, \varphi = b\, D_c\, \mathcal{F}\, \varphi.$$

(b) [6p] State the appropriate definition of the operator $D_a : \mathcal{S}^* \to \mathcal{S}^*$, and derive for $T \in \mathcal{S}^*$ a formula for $\mathcal{F} D_a T$ analogous to (EQ1). Be careful in motivating your work!

(c) [6p] Fix a function $h \in C_{\text{b}}(\mathbb{R})$ (*i.e.* h is bounded and continuous), and set $f_n = D_{1/n}h$ for $n = 1, 2, 3, \ldots$. Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges in $\mathcal{S}^*$ and give the limit.

(d) [6p] With f_n as in (c), set $\hat{f}_n = \mathcal{F} f_n$. Does the sequence $(\hat{f}_n)_{n=1}^{\infty}$ converge in $\mathcal{S}^*$? If so, to what?

(e*) [2p] Give an example of a distribution $h \in \mathcal{S}^*$ such that $(D_{1/n}h)_{n=1}^{\infty}$ does not converge in $\mathcal{S}^*$.

Problem 3: (25 points)

(a) [5p] For d a positive integer, and s a real number, define the Sobolev space $H^s(\mathbb{R}^d)$.

(b) [5p] For which s, if any, is it necessarily the case that all functions in $H^s(\mathbb{R}^d)$ are continuous?

(c) [10p] Let $f \in L^2(\mathbb{R})$. Show that the equation $-u'' + u = f$ has a unique solution $u \in H^2(\mathbb{R})$.

(d*) [5p] Give an example of a function $f \in L^2(\mathbb{R}^2)$ such that the equation

$$-\frac{\partial^2 u}{\partial x_1^2} + u = f,$$

does not have a solution in $H^2(\mathbb{R}^2)$.

Problem 4: (12 points)

(a) [4p] State the definition of a *measure*.

(b) [4p] Let $(X, \mathcal{T})$ be a topological space. State the definition of the *Borel σ-algebra* associated with $(X, \mathcal{T})$.

(c) [4p] State the definition of the *essential supremum*.

Problem 5: (12 points) Let $\mathbb{N}$ denote the set of positive integers, and let $\mathcal{A}$ denote the collection of all subsets of $\mathbb{N}$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of real numbers, and define a function

$$\mu : \mathcal{A} \to \mathbb{R} : \Omega \mapsto \sum_{n \in \Omega} \alpha_n.$$

Under what conditions on the numbers (α_n) is μ a measure? Is it ever a finite measure? Is it ever a σ-finite measure? No motivation required.