Applied Analysis (APPM 5450): Midterm 2 — solutions

8.30am – 9.50am, March 15, 2010. Closed books.

Problem 1: (30 points) Let A be a bounded linear operator on a Hilbert space H.

(a) (10 points) Suppose that $\lambda \in \sigma_{\rm p}(A)$. Prove that $\bar{\lambda} \in \sigma(A^*)$. Can you tell what part of the spectrum $\bar{\lambda}$ belongs to?

(b) (10 points) Suppose that A is self-adjoint, and that M is an invariant subspace of A. Prove that $M^\perp$ is also an invariant subspace of A.

(c) (10 points) Suppose that A is compact and self-adjoint. Which statements are necessarily true?

(i) $\sigma(A) \subseteq \mathbb{R}$.

(ii) $\sigma_{\rm r}(A) = \emptyset$.

(iii) $\sigma_{\rm c}(A) = \emptyset$.

(iv) $\sigma(A) \subseteq \big(\sigma_{\rm p}(A) \cup \{0\}\big)$.

(v) $\sigma(A)$ contains infinitely many points.

(vi) If $\lambda \neq 0$, then $\dim(\ker(A - \lambda\, I)) < \infty$.

No motivation required.

Solution:

(a) Since $\lambda \in \sigma_{\rm p}(A)$, we know that $A - \lambda I$ has a non-trivial null-space. It follows that $A^* - \bar{\lambda} I$ cannot be onto since

$$\overline{\operatorname{ran}(A^* - \bar{\lambda} I)} = \big(\ker(A - \lambda I)\big)^\perp. \qquad (1)$$

Therefore $A^* - \bar{\lambda} I$ cannot be invertible, and so $\bar{\lambda} \in \sigma(A^*)$. As for the part of the spectrum, (1) rules out the possibility that $\operatorname{ran}(A^* - \bar{\lambda} I)$ is dense, so $\bar{\lambda} \notin \sigma_{\rm c}(A^*)$. Answer: $\bar{\lambda} \in \sigma_{\rm p}(A^*)$ or $\bar{\lambda} \in \sigma_{\rm r}(A^*)$.

(b) Fix $x \in M^\perp$. We need to prove that $A\,x \in M^\perp$. For any $y \in M$, we have

$$(y,\, Ax) = (A\,y,\, x) = 0$$

where the second equality follows from the fact that $A\,y \in M$ (since M is invariant) and $x \in M^\perp$. Since $(y,\, A\,x) = 0$ for all $y \in M$, it follows that $A\,x \in M^\perp$.

(c) (i), (ii), (iv), and (vi) are true.

Problem 2: (20 points)

(a) (6 points) Define what is meant by the *derivative* of a distribution $T \in \mathcal{S}^*(\mathbb{R})$.

(b) (14 points) Define $f \in \mathcal{S}^*(\mathbb{R})$ via $f(x) = |x|$. Calculate the distributional derivatives f' and f''. Please motivate carefully.

Solution:

(a) The derivative T' is the map $T' : \mathcal{S} \to \mathbb{C} : \varphi \mapsto -T(\varphi')$.

(b) Let $\varphi \in \mathcal{S}$. Then

$$\begin{aligned}\langle f', \varphi \rangle = -\langle f, \varphi' \rangle = -\int_{-\infty}^{0} (-x)\varphi'(x)\,dx - \int_{0}^{\infty} x\varphi'(x)\,dx \\ = [x\varphi(x)]_{-\infty}^{0} - \int_{-\infty}^{0} \varphi(x)\,dx - [x\varphi(x)]_{0}^{\infty} + \int_{-\infty}^{0} \varphi(x)\,dx.\end{aligned}$$

Now observe that $[x\varphi(x)]_{-\infty}^{0} = 0 \cdot \varphi(0) - \lim_{t\to\infty} t\,\varphi(t) = -\lim_{t\to\infty} t\,\varphi(t)$. The limit is zero since φ decays faster than any polynomial. Analogously, $[x\varphi(x)]_{0}^{\infty} = 0$. It follows that

$$\langle f', \varphi \rangle = -\int_{-\infty}^{0} \varphi(x)\,dx + \int_{-\infty}^{0} \varphi(x)\,dx = \langle g, \varphi \rangle,$$

provided that we define the function g via

$$g(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

So $f' = g$. (Note that the value of $g(0)$ is irrelevant, any finite value can be assigned.) Furthermore,

$$\begin{aligned}\langle f'', \varphi \rangle = \langle g', \varphi \rangle = -\langle g, \varphi' \rangle = \int_{-\infty}^{0} \varphi'(x)\,dx - \int_{0}^{\infty} \varphi'(x)\,dx \\ = [\varphi(x)]_{-\infty}^{0} - [\varphi(x)]_{0}^{\infty} = \varphi(0) - (-\varphi(0)) = 2\varphi(0) = \langle 2\delta, \varphi \rangle,\end{aligned}$$

so $f'' = 2\delta$.

Problem 3: (20 points) Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ denote the Schwartz space over $\mathbb{R}$.

(a) (6 points) Define what it means for a sequence to converge in $\mathcal{S}$. If your definition relies on any norms, semi-norms, metrics, bases, *etc*, then state the definition of these.

(b) (8 points) Let α be a positive integer. Prove that $\left(\frac{d}{dx}\right)^\alpha : \mathcal{S} \to \mathcal{S}$ is a continuous map.

(c) (6 points) Set $\varphi_n(x) = e^{-(x-n)^2}$. Does the sequence $(\varphi_n)_{n=1}^\infty$ converge in $\mathcal{S}$? If so, to what?

Solution:

(a) For $k, \alpha \in \mathbb{Z}_+$, set

$$||\varphi||_{\alpha,k} = \sup_{x\in\mathbb{R}} (1+|x|^2)^{k/2} |\varphi^{(\alpha)}(x)|,$$

where $\varphi^{(\alpha)}$ denotes the α derivative of φ. Then

$$\varphi_n \to \varphi \text{ in } \mathcal{S} \qquad \Leftrightarrow \qquad \text{For every } \alpha, k \text{ we have } \lim_{n\to\infty} ||\varphi_n - \varphi||_{\alpha,k} = 0.$$

(b) Fix α. Suppose $\varphi_n \to \varphi$ in $\mathcal{S}$. This is to say that

(2) $$\text{For every } \alpha, k \text{ we have } \lim_{n\to\infty} ||\varphi_n - \varphi||_{\alpha,k} = 0.$$

We need to prove that $\varphi_n^{(\alpha)} \to \varphi^{(\alpha)}$ in $\mathcal{S}$. Fix k, β. Then

$$||\varphi_n^{(\alpha)} - \varphi^{(\alpha)}||_{k,\beta} = \sup_{x\in\mathbb{R}} (1+|x|^2)^{k/2} |\varphi_n^{(\alpha+\beta)}(x) - \varphi^{(\alpha+\beta)}(x)| = ||\varphi_n - \varphi||_{k,\alpha+\beta}.$$

By (2) we find that

$$\lim_{n\to\infty} ||\varphi_n^{(\alpha)} - \varphi^{(\alpha)}||_{k,\beta} = 0.$$

Since k, β were arbitrary, this proves that $\varphi_n^{(\alpha)} \to \varphi^{(\alpha)}$ in $\mathcal{S}$.

(c) Since $(\varphi_n)_{n=1}^\infty$ converges pointwise to the zero-function, the only possible limit would be the zero function. But for any n, we have

$$||\varphi_n - 0||_{0,0} = \sup_x |\varphi_n(x)| = 1.$$

It follows that $(\varphi_n)_{n=1}^\infty$ cannot converge.

Problem 4: (30 points) Let H be a Hilbert space with an orthonormal basis $(\varphi_n)_{n=1}^{\infty}$. Consider the operators

$$A_N\, x = \sum_{n=1}^{N} \frac{1}{n} (\varphi_n,\, x)\, \varphi_n, \qquad \text{and} \qquad B_N\, x = \exp(iA_N) = \sum_{n=1}^{N} e^{i/n} (\varphi_n,\, x)\, \varphi_n.$$

The sequences $(A_N)_{N=1}^{\infty}$ and $(B_N)_{N=1}^{\infty}$ have the strong limits A and B, respectively.

(a) (10 points) Put a check-mark in all the boxes that are correct (no motivation required):

	Compact	Self-adjoint	Skew-adjoint	Normal	Unitary	One-to-one	Onto
A_N	T	T		T			
A	T	T		T			
B_N	T			T			
B				T	T	T	T

(b) (10 points) Do either of the sequences $(A_N)_{N=1}^{\infty}$ or $(B_N)_{N=1}^{\infty}$ converge in norm? Motivate your answers.

(c) (10 points) Specify the spectra of A and B and identify their different parts (*i.e.* specify σ_p, $\sigma_{\rm c}$, and $\sigma_{\rm r}$). No motivation required.

Solution:

(b) $(A_N)_{N=1}^{\infty}$ does converge in norm: Let $x \in H$. Then

$$||(A - A_N)x||^2 = \left|\left| \sum_{n=N+1}^{\infty} \frac{1}{n} (\varphi_n\, x)\, \varphi_n) \right|\right|^2 = \{\text{Pythagoras}\} = \sum_{n=N+1}^{\infty} \left| \frac{1}{n} (\varphi_n\, x) \right|^2$$

$$\leq \frac{1}{(N+1)^2} \sum_{n=N+1}^{\infty} |(\varphi_n\, x)|^2 \leq \frac{1}{(N+1)^2} ||x||^2.$$

It follows that $||A - A_N|| \leq 1/(N+1)$ so $A_N \to A$ in norm.

$(B_N)_{N=1}^{\infty}$ does not converge in norm: We have

$$||B - B_N|| \geq ||(B - B_N)\, \varphi_{N+1}|| = ||e^{i/(N+1)} \varphi_{N+1}|| = |e^{i/(N+1)}| = 1.$$

(c)

$$\sigma_{\rm p}(A) = \{1/n\}_{n=1}^{\infty}, \qquad \sigma_{\rm c}(A) = \{0\}, \qquad \sigma_{\rm r}(A) = \emptyset.$$

$$\sigma_{\rm p}(A) = \{e^{i/n}\}_{n=1}^{\infty}, \qquad \sigma_{\rm c}(A) = \{1\}, \qquad \sigma_{\rm r}(A) = \emptyset.$$